

# NEW COMBINATORIAL TECHNIQUES FOR NONLINEAR ORDERS

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# NEW COMBINATORIAL TECHNIQUES FOR NONLINEAR ORDERS

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*To my grandfather,*

*Edward L. Kaufman.*

*May my successes always serve to honor your memory.*

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## SUMMARY

Extremal combinatorics has developed into a rich area of mathematical research with many important and deep results. Many of the techniques have been equally useful in understanding other areas of mathematics as well. This thesis focuses on the use of extremal techniques in analyzing problems that historically have been associated with other areas of discrete mathematics. We establish new techniques for analyzing combinatorial problems with two different types of nonlinear orders, and then use them to solve important previously-open problems in mathematics. In addition, we use entropy techniques to establish a variety of bounds in the theory of sumsets.

In the second chapter, we examine a problem of Füredi and Hajnal regarding forbidden patterns in  $(0,1)$ -matrices [26]. We introduce a new technique that gives an asymptotically tight bound on the number of 1-entries that a  $(0,1)$ -matrix contain while avoiding a fixed permutation matrix. We use this result to solve the Stanley–Wilf conjecture, a well-studied open problem in enumerative combinatorics [4, 11, 9, 10, 56]. Furthermore, we generalize the technique and give a generalized result on  $d$ -dimensional matrices.

In the third chapter, we examine a problem of Pinchasi and Radoičić first posed in [46] by developing a new technique for analyzing cyclically ordered sets. We prove an upper bound on the sizes of such sets, given that their orders have the *intersection reverse* property. We then use this to give an upper bound on the number of edges that a graph on  $n$  vertices can have, assuming that the graph can be drawn in such a way that no cycle of length four has intersecting edges. This improves the previously best known bound and (up to a log-factor) matches the best known lower bound. This result, in turn, implies improved bounds on a number of well-studied and important problems in geometric combinatorics, most notably the complexity of pseudo-circle arrangements [2, 5, 16, 46].

In the final chapter, we use entropy techniques to establish new bounds in the theory of sumsets. In particular, we show that such sets behave fractionally submultiplicatively,



which in turn provides a vast number of new Plunecke-type inequalities of the form first introduced by Gyarmati, Matolcsi, and Ruzsa [28].

# CHAPTER I

## INTRODUCTION

We begin by discussing the general overview of the thesis and list the contributions made to the subject of combinatorics. Following this, the basic concepts that we use throughout the thesis are presented, as well as recognition of coauthors and sponsors.

### *1.1 Contributions*

Due to the disjoint nature of the topics presented in this thesis, much of the history will be discussed within the respective chapters. However, we will discuss some of the implications here.

The results in Chapter 2 establish new tools for problems in permutation avoidance. The most significant contribution is the solving of the Stanley–Wilf conjecture, which was considered by many to be the most interesting open problem in pattern avoidance. The results in this chapter are actually far stronger than the original conjecture, and the simplicity of the proofs is particularly remarkable. The reader should reference Section 2.1.2 and Section 2.3.2 for the history of the problems.

The results of this chapter were originally published in [42] and [37], and since publication there have been a number of papers that use the techniques and results. The most interesting use of the techniques can be seen in a paper by Balogh, Bollobás, and Morris [8]. They extend the ideas of permutation avoidance in graphs to permutations in hypergraphs by considering (hypergraph) incidence matrices rather than the adjacency matrices that are used primarily in this work. Doing so requires them to avoid groups of patterns (since the edge set can appear in a number of orders in the incidence matrix), but they are able to adapt the techniques in [42] for their purposes.

Perhaps the most interesting corollary of the results in Chapter 2, however, comes from a paper by Arthur [6] regarding the complexity of sorting algorithms. The worst case lower bound for sorting an arbitrary list of length  $n$  is known to be  $\Omega(n \log n)$ ; however given

further information about an unsorted list, one can hope to lower this significantly. Arthur shows that permutation avoidance can be such a piece of information. In particular, he uses the techniques of this chapter to show that there exists a collection  $\Sigma$  of permutations and (respectively) algorithms  $\{\mathcal{A}_\sigma\}_{\sigma \in \Sigma}$  such that given  $\pi$ , an unsorted list of length  $n$  which avoids  $\sigma \in \Sigma$ ,  $\mathcal{A}_\sigma$  can sort  $\pi$  in  $O(n \log \log \log n)$  steps. Not only does this improve on the established lower bound in the general case, it begins to come remarkably close to a linear algorithm (which is the best one can hope for).

In Chapter 3, we bound the number of edges that a graph can have if there exists a drawing of the graph on the plane without self-intersecting  $C_4$  subgraph. This was known to be useful in a number of geometric problems, and was widely studied in the geometric combinatorics community (i.e., [2, 5, 46]). The most notable contribution of the work in this chapter, originally published in [43], is reducing the known upper bound on the problem to match the lower bound up to a log factor (see Section 3.1.2 for details). However, a number of other insights and improvements are made. In Section 3.2, we consider a number of the known consequences of our initial result, and in some cases we are able to improve upon these results even beyond what the new bounds can give. For example, we are able to extend the bound on  $x$ -monotone pseudo-circles to arbitrary planar pseudo-circles and then even further to spherical collections of pseudo-circles.

Finally, in Chapter 4, we examine a well-studied problem in additive combinatorics. The idea of bounding the growth of sumsets (see the beginning of the chapter for definitions) has appeared in numerous contexts, including the celebrated paper of Green and Tao that proves the existence of arbitrarily long arithmetic progressions in the primes [27].

The general sumset problem considers an Abelian group  $\mathcal{A}$  and a collection of subsets of the group elements  $S_1, \dots, S_k$ . The *sumset* of these subsets is  $S = \{s_1 + s_2 + \dots + s_k : s_i \in S_i\}$ , and the goal is to bound the size of  $S$  using knowledge about the factor sets  $S_i$ . Numerous bounds appeared recently in the literature (see Section 4.1.2 for details), each of which seems to suggest that sumsets behave in a sub-multiplicative manner, like the much more well-studied entropy function. In this chapter, we show that this is in fact true, proving a number of conjectures by various authors. Some of the ideas in this chapter

were discovered independently by Balister and Bollobás [7], who used the results of [41] to develop a hierarchy of entropy inequalities. Some of our methods, however, are completely new and can be used to extend the results in [7] beyond sumsets in ways that were not considered in that paper.

## 1.2 Preliminaries

Definitions that are more specific to the results will be given in the appropriate sections. We will use the following two conventions: we will use the abbreviated notation  $[n]$  to denote the set  $\{1, \dots, n\}$ , and we will consider all log functions in this paper are considered to be binary (base 2).

### 1.2.1 Functions

Given a function  $f : A \rightarrow B$  and  $X \subseteq A$ , we will write  $f(X)$  to denote  $\{f(i) : i \in X\}$ . If the elements of  $A$  and  $B$  are linearly ordered, we will say that  $f$  is *order preserving* if for all  $x, y \in A$ ,  $x < y$  if and only if  $f(x) < f(y)$ . It should be clear that any such  $f$  must be an injection. Furthermore, we will use the following notations when discussing the asymptotic growth of a function  $g(x)$ :

$$f = O(g) \quad \Rightarrow \quad \text{there exists a constant } C_1 \text{ s.t. } f(x) \leq C_1 \cdot g(x) \text{ for all } x \in \mathbb{R}$$

$$f = \Omega(g) \quad \Rightarrow \quad \text{there exists a constant } C_2 \text{ s.t. } f(x) \geq C_2 \cdot g(x) \text{ for all } x \in \mathbb{R}$$

$$f = \Theta(g) \quad \Rightarrow \quad f = O(g) \text{ and } f = \Omega(g)$$

The constants  $C_1$  and  $C_2$  that are implied by the asymptotic notation will be referred to as *hidden constants*.

The *Ackermann function*  $A(x, y)$  is defined recursively as follows:

$$A(x, y) = \begin{cases} y + 1 & \text{if } x = 0, \\ A(x - 1, 1) & \text{if } y = 0, \\ A(x - 1, A(x, y - 1)) & \text{otherwise} \end{cases}$$

We will not use the Ackermann function directly in this paper; however it should be noted that  $A(n, n)$  grows *extremely* quickly as a function of  $n$  (so quickly in fact, that it is not

primitive recursive). The more relevant function for our purpose is the *inverse Ackermann function*  $\alpha(n) = A^{-1}(n, n)$ . The two properties of  $\alpha(n)$  that will be important in our context are that it grows *extremely* slowly and that it is unbounded. Thus if  $f = O(\alpha(n))$ , then  $f = O(g(n))$  for every unbounded primitive-recursive function  $g$ , but it is *not* necessarily the case that  $f = O(1)$ . See [49] for more details.

### 1.2.2 Graphs

A *graph*  $G = (V, E)$  consists of a finite set  $V = V(G)$  and a finite collection  $E = E(G)$  of distinct unordered pairs of distinct elements of  $V$ .<sup>1</sup> The elements of  $V$  are called *vertices* and the elements of  $E$  are called *edges*. If  $e = \{u, v\}$  is an edge, we will say that vertices  $u$  and  $v$  are adjacent and denote this as  $u \sim v$  or  $v \sim u$ . The *neighborhood of  $v$*  is  $N(v) = \{u : u \sim v\}$ , the collection of all vertices adjacent to  $v$ . Finally, we will say that the *degree* of a vertex  $u$  (denoted  $\deg(u)$ ) is  $|N(u)|$ , the number of vertices  $v \in V$  such that  $v \sim u$ .

A *subgraph* of  $G = (V, E)$  is another graph  $K(U, F)$  such that  $U \subseteq V$  and  $F \subseteq E$ , and for every  $f \in F$ ,  $f \subseteq U$ . For  $X \subseteq V(G)$ , we write  $G[X]$  to be the subgraph *induced by  $X$* ; that is, the subgraph of  $G$  with vertex set  $X$  and such that  $F$  contains every edge in  $G$  that lies completely within the vertex set  $X$ . A graph  $G = (V, E)$  is called *bipartite* if  $V$  can be partitioned into two sets  $X, Y$  such that neither  $G[X]$  nor  $G[Y]$  have any edges.

### 1.2.3 Hypergraphs

A *hypergraph*  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$  consists of a finite set  $\mathcal{V}$  and a finite collection  $\mathcal{E}$  of distinct *subsets* of  $\mathcal{V}$ . All of the definitions used for graphs extend to hypergraphs, the only difference being that edges can be of variable sizes. If a hypergraph has all edges containing exactly  $d$  vertices, we will call it *d-uniform*; for example, a 2-uniform hypergraph is a graph. A hypergraph is called *t-partite* if there exists a partition  $\mathcal{V}_1, \dots, \mathcal{V}_t$  of  $\mathcal{V}$  such that  $|E \cap \mathcal{V}_i| \leq 1$  for all  $E \in \mathcal{E}$  and all  $i \in [t]$ . Note that this reduces to the bipartite condition when  $G$  is a graph and  $t = 2$ .

The *order*  $v(\mathcal{H})$  of  $\mathcal{H}$  is the number of vertices  $v(\mathcal{H}) = |\mathcal{V}(\mathcal{H})|$ , the *size*  $e(\mathcal{H})$  is the number

---

<sup>1</sup>In some circles, this is known as a *simple graph*, as we have disallowed multiple copies of the same pair (parallel edges) and pairs with the same element twice (loops)

of edges  $e(\mathcal{H}) = |\mathcal{E}(\mathcal{H})|$ , and the *weight*  $i(\mathcal{H})$  is the number of incidences  $i(\mathcal{H}) = \sum_{E \in \mathcal{E}} |E|$ . Given an edge  $E$ , we define the operation of *shrinking* as replacing  $E$  with  $E \cap X$  for some  $X \subset \mathcal{V}$ .

For simplicity we do not allow isolated vertices, unlike in the graph case; for our extremal problems this restriction is immaterial, as isolated vertices in this case can be represented by singleton edges. In general we *do* allow multiple edges, and will denote a hypergraph as *simple* if it has no multiple edges.

### 1.2.4 Matrices and $d$ -dimensional Matrices

To accentuate that all matrices and  $d$ -dimensional matrices will be binary—that is, the entries will all be elements in  $\{0, 1\}$ —we will refer to them as  $(0, 1)$ -matrices. We will not be concerned with the algebraic properties of matrices, rather they are a more natural way to envision order relations in graphs and hypergraphs.

Given a graph  $G = (V, E)$ , the *adjacency matrix* of  $G$  is the  $|V| \times |V|$   $(0, 1)$ -matrix  $A$  such that  $A_{i,j} = 1$  if and only if  $\{i, j\} \in E$ . If  $G$  is bipartite with parts  $X, Y \subset V$ , then we will generally use the *bipartite adjacency matrix* of  $G$ : the  $|X| \times |Y|$   $(0, 1)$ -matrix  $B$  such that  $B_{i,j} = 1$  if and only if  $\{i, j\} \in E$ . The *incidence matrix* of  $G$  is the  $|V| \times |E|$   $(0, 1)$ -matrix  $M$  such that  $M_{i,j} = 1$  if and only if vertex  $v_i$  is incident to edge  $e_j$ .

## 1.3 Coauthors and Sponsors

Chapter 2 consists of a merging of two papers. The first paper [42] consists of work done with Gábor Tardos while the author was a visitor at the Rényi Institute in Budapest, Hungary under the support of a Fulbright Fellowship to Hungary. The second paper [37] consists of work done with Martin Klazar while the author was a visitor at Charles University in Prague, Czech Republic under the support of funds from DIMATIA. The actual writing of [37] occurred later while the author was supported by a VIGRE Fellowship at the Georgia Institute of Technology. Chapter 3 consists of work that was done with Gábor Tardos, again while the author was supported by a Fulbright Fellowship in Budapest, Hungary, and represents the results originally published in [43]. Finally, Chapter 4 consists of work done with Prasad Tetali while the author was a graduate student at Georgia Tech. This work, as

well as the writing of the thesis, was done under the support of an NSF Graduate Research Fellowship.

## CHAPTER II

### PERMUTATION AVOIDANCE

This chapter considers problems in pattern avoidance, concentrating on the situation when the pattern is a permutation (in whatever structure we are considering). We first settle some conjectures in the more popular areas of pattern avoidance, and then extend these results to more complicated structures. It should be noted that, in order to avoid unnecessary complications in the asymptotic analysis, we make no effort to optimize any of the constants in this chapter.

#### 2.1 Introduction

This section settles three related conjectures concerning more popular areas of pattern avoidance. To state the conjectures we define the term “avoiding” in several contexts.

##### 2.1.1 Preliminaries

A *permutation*  $\pi$  of a set  $S$  is a linear order of the elements of the set. The collection of permutations of the set  $[n]$  will be denoted  $S_n$ .

A  $(0,1)$ -matrix  $P$  is called a *permutation matrix*  $P$  if every row of  $P$  and every column of  $P$  has a single 1-entry. Note that there is a bijection between the permutations in  $S_n$  and the  $n \times n$  permutation matrices, defined by

$$P(\pi)_{i,j} = \begin{cases} 1 & \text{if } j = \pi(i) \\ 0 & \text{otherwise} \end{cases}$$

for  $\pi \in S_n$ .

##### 2.1.1.1 Permutation avoidance

Given  $\pi \in S_n$  and  $\sigma \in S_k$ , we will say that  $\pi$  *contains*  $\sigma$  if there is an order preserving function  $f : [k] \rightarrow [n]$  such that  $\sigma(i) < \sigma(j)$  if and only if  $\pi(f(i)) < \pi(f(j))$ . If, on the other hand, no such  $f$  exists, then we will say that  $\pi$  *avoids*  $\sigma$ .



For example, let  $\pi = (2, \underline{1}, 5, \underline{3}, \underline{4})$  and  $\sigma = (1, 2, 3)$ . To see that  $\pi$  contains  $\sigma$ , define  $f$  as  $f(1) = 2, f(2) = 4, f(3) = 5$  (the underlined elements). Thus  $f$  is order preserving and satisfies the definition of containment.

For a permutation  $\pi$  let  $S_n(\pi)$  be the number of  $n$ -permutations avoiding  $\pi$ .

### 2.1.1.2 Sequence avoidance

The collection of all finite sequences of elements in  $[n]$  will be denoted  $[n]^*$ . A sequence  $(a_1, a_2, \dots, a_m) \in [n]^*$  is called *k-sparse* if, for all  $i < j$ ,  $a_i = a_j$  implies  $j - i \geq k$  (that is, repeated symbols always occur at distance  $\geq k$ ). Much like in permutations, a sequence  $a = (a_1, a_2, \dots, a_h) \in [n]^*$  is said to *contain* the sequence  $b = (b_1, b_2, \dots, b_k) \in [m]^*$  if there exists an order preserving function  $f : [k] \rightarrow [h]$  such that  $b_{f(i)} < b_{f(j)}$  if and only if  $a_i < a_j$ . Otherwise we say that  $a$  *avoids*  $b$ .

For a sequence  $a$ , let  $l_k(a, n)$  be the length of the longest  $k$ -sparse sequence in  $[n]^*$  that avoids  $a$ . Note the importance of the  $k$ -sparse constraint, as otherwise arbitrarily long sequences can be formed by repeating the same entry (assuming  $a$  contains more than one distinct element).

### 2.1.1.3 (0,1)-matrix avoidance

Let  $A$  be an  $m \times n$  (0,1)-matrix and  $B = (b_{ij})$  be a  $k \times l$  (0,1)-matrix. We say that  $A$  *contains*  $B$  if there exists a  $k \times l$  submatrix  $D = (d_{ij})$  of  $A$  such that  $d_{ij} = 1$  whenever  $b_{ij} = 1$ . Otherwise we say that  $A$  *avoids*  $B$ . Notice that we can delete rows and columns of  $A$  and change 1-entries to 0-entries (to obtain  $B$  exactly) but we cannot permute the remaining rows and columns or change 0-entries to 1-entries. If  $A$  contains  $B$  we identify the 1-entries of the matrix  $A$  corresponding to the entries  $d_{ij}$  of  $D$  with  $b_{ij} = 1$  and say that these entries of  $A$  *represent*  $B$ .

For a (0,1) matrix  $A$  let  $f(n, A)$  be the maximum number of 1-entries in an  $n \times n$  (0,1) matrix avoiding  $A$ .

One should notice that the three definitions of avoidance are intimately related. In particular, every permutation is a sequence and every sequence can be written as a (0,1)-matrix (in an identical way that the bijection between permutations and permutation matrices was

formed at the beginning of this section).

### 2.1.2 History

The earliest recorded attempts to find the asymptotic behavior of  $S_n(\pi)$  date back to the 1990 Ph.D. thesis of Julian West [55] (though some claim the problem is even older [11]). His Question 3.4.3 is more specific; he asks if  $S_n(\pi)$  and  $S_n(\pi')$  are asymptotically equal for  $k$ -permutations  $\pi$  and  $\pi'$ . Slightly afterward, Richard Stanley and Herbert Wilf offered the following somewhat more cautious conjecture:

**Conjecture 2.1.1.** *For all permutations  $\pi$  there exists a constant  $c = c_\pi$  such that  $S_n(\pi) \leq c^n$ .*

The conjecture, dubbed the Stanley–Wilf conjecture, quickly grew a strong following. In 1997, Miklós Bóna [9] showed that West’s original conjecture was too strong by finding 4-permutations  $\pi$  and  $\pi'$  with  $S_n(\pi)$  and  $S_n(\pi')$  displaying different growth rates. However, little else had been established.

Soon afterward, several special cases of Conjecture 2.1.1 were shown to be true. Most notably was the result of Bóna in [10], where he proved the Stanley–Wilf conjecture for *layered* permutations  $\pi$ , that is, for permutations consisting of an arbitrary number of increasing blocks with all elements of a block smaller than the elements of the previous block. Alon and Friedgut [4] then proved the conjecture for permutations consisting of an increasing sequence followed by a decreasing one or vice versa.

Increasingly tight asymptotic bounds had also been established. Using a result of Klazar [32] on generalized Davenport–Schinzel sequences Alon and Friedgut [4] showed approximate versions of the Stanley–Wilf conjecture where the exponential bound was replaced by  $2^{O(n\gamma(n))}$ , where  $\gamma$  is an extremely slow growing function related to the inverse Ackermann function. Their study was inspired by a related question, that has since become known as the Alon–Friedgut conjecture:

**Conjecture 2.1.2.** *Let  $\sigma$  be a permutation of size  $k$ , and let  $a = (\sigma(1), \sigma(2), \dots, \sigma(k)) \in [k]^*$ . Then  $l_k(a, n) = O(n)$ .*

Later that year, Klazar showed a link between the preceding conjectures and a problem stated independently in [26], a paper of Zoltan Füredi and Péter Hajnal that began a systematic study of avoidance problems on  $(0,1)$ -matrices. In [35], Klazar showed that the following question from [26], which he named the Füredi–Hajnal conjecture, implied both the Stanley–Wilf and Alon–Friedgut conjectures.

**Conjecture 2.1.3.** *For all permutation matrices  $P$  we have  $f(n, P) = O(n)$ .*

In the following section, we give a surprisingly simple and straightforward proof of Conjecture 2.1.3 and then reproduce the results of Klazar that show how it implies both Conjecture 2.1.1 and Conjecture 2.1.2. We then discuss a host of other problems that are directly implied by our result. In Section 2.4 we generalize Conjecture 2.1.3 to  $d$  dimensions, replacing matrices with  $d$ -ary relations on  $[n]$ , and then prove the appropriate extension.

## 2.2 *Permutation Matrix Avoidance*

Conjecture 2.1.3 is proved by establishing a linear recursion for  $f(n, P)$  in Lemma 2.2.4, that in turn is based on three rather simple lemmas. We partition the larger matrix into *blocks*. This idea appears in several related papers, e. g. in [35], but we use larger blocks than were previously considered and introduce a new technique that allows for tighter analysis.

### 2.2.1 *Proof of the Füredi–Hajnal Conjecture*

Throughout these lemmas, we let  $P$  be a fixed  $k \times k$  permutation matrix and  $A$  be an  $n \times n$  matrix with  $f(n, P)$  1-entries that avoids  $P$ . For simplicity, we assume  $k^2$  divides  $n$  (we will correct for this assumption in the proof of Theorem 2.2.5) and define  $S_{ij}$  to be the square submatrix of  $A$  consisting of the entries  $a_{i'j'}$  with  $i' \in [k^2(i-1)+1, k^2i]$ ,  $j' \in [k^2(j-1)+1, k^2j]$ . We define the *reduction*  $B = (b_{ij})$  to be the  $(n/k^2) \times (n/k^2)$   $(0,1)$ -matrix with  $b_{ij} = 0$  if and only if all entries of  $S_{ij}$  are zero. We say that a block is *wide* (respectively *tall*) if it contains 1-entries in at least  $k$  different columns (respectively rows).

**Lemma 2.2.1.**  *$B$  avoids  $P$ .*

*Proof.* Assume not and consider the  $k$  1-entries of  $B$  representing  $P$ . Choose an arbitrary 1-entry from the  $k$  corresponding blocks of  $A$ . They represent  $P$ , contradicting the fact that  $A$  avoids  $P$ .  $\square$

**Lemma 2.2.2.** *Consider the set (column) of blocks  $C_j = \{S_{ij} : i = 1, \dots, \frac{n}{k^2}\}$ . The number of blocks in  $C_j$  that are wide is less than  $k \binom{k^2}{k}$ .*

*Proof.* Assume not. By the pigeonhole principle, there exist  $k$  blocks in  $C_j$  that have a 1-entry in the same columns  $c_1 < c_2 < \dots < c_k$ . Let  $S_{d_1j}, \dots, S_{d_kj}$  be these blocks with  $1 \leq d_1 < d_2 < \dots < d_k \leq n/k^2$ . For each 1-entry  $p_{rs}$ , pick any 1-entry in column  $c_s$  of  $S_{d_rj}$ . These entries of  $A$  represent  $P$ , a contradiction.  $\square$

**Lemma 2.2.3.** *Consider the set (row) of blocks  $R_i = \{S_{ij} : j = 1, \dots, \frac{n}{k^2}\}$ . The number of blocks in  $R_i$  that are tall is less than  $k \binom{k^2}{k}$ .*

*Proof.* The same proof applies as for Lemma 2.2.2.  $\square$

With these tools, the main lemma follows:

**Lemma 2.2.4.** *For a  $k \times k$  permutation matrix  $P$  and  $n$  divisible by  $k^2$  we have*

$$f(n, P) \leq (k-1)^2 f\left(\frac{n}{k^2}, P\right) + 2k^3 \binom{k^2}{k} n.$$

*Proof.* We consider three types of blocks:

- $X_1 = \{ \text{blocks that are wide} \}$ .

$$|X_1| \leq \frac{n}{k^2} k \binom{k^2}{k} \text{ by Lemma 2.2.2.}$$

- $X_2 = \{ \text{blocks that are tall} \}$ .

$$|X_2| \leq \frac{n}{k^2} k \binom{k^2}{k} \text{ by Lemma 2.2.3.}$$

- $X_3 = \{ \text{nonempty blocks that are neither wide nor tall} \}$ .

$$|X_3| \leq f\left(\frac{n}{k^2}, P\right) \text{ by Lemma 2.2.1.}$$

This includes all of the nonempty blocks. We bound  $f(n, P)$ , the number of ones in  $A$ , by summing estimates of the number of ones in these three categories of blocks. Any block contains at most  $k^4$  1-entries and a block of  $X_3$  contains at most  $(k-1)^2$  1-entries. Thus,

$$\begin{aligned} f(n, P) &\leq k^4|X_1| + k^4|X_2| + (k-1)^2|X_3| \\ &\leq 2k^3 \binom{k^2}{k} n + (k-1)^2 f\left(\frac{n}{k^2}, P\right). \end{aligned}$$

as required.  $\square$

Solving the linear recursion above gives the following theorem, which in turn proves Conjecture 2.1.3.

**Theorem 2.2.5.** *For a  $k \times k$  permutation matrix  $P$  we have*

$$f(n, P) \leq 2k^4 \binom{k^2}{k} n.$$

*Proof.* We proceed by induction on  $n$ . The base cases (when  $n \leq k^2$ ) are trivial. Now assume the hypothesis to be true for all  $n < n_0$  and consider the case  $n = n_0$ . We let  $n'$  be the largest integer less than or equal to  $n$  which is divisible by  $k^2$ . Then by Lemma 2.2.4, we have:

$$\begin{aligned} f(n, P) &\leq f(n', P) + 2k^2 n \\ &\leq (k-1)^2 f\left(\frac{n'}{k^2}, P\right) + 2k^3 \binom{k^2}{k} n' + 2k^2 n \\ &\leq (k-1)^2 \left[ 2k^4 \binom{k^2}{k} \frac{n'}{k^2} \right] + 2k^3 \binom{k^2}{k} n' + 2k^2 n \\ &\leq 2k^2 ((k-1)^2 + k + 1) \binom{k^2}{k} n \\ &\leq 2k^4 \binom{k^2}{k} n \end{aligned}$$

where the last inequality is true for all  $k \geq 2$ .  $\square$

### 2.2.2 Proof of the Stanley–Wilf Conjecture

For a  $(0,1)$ -matrix  $M$  let  $T_n(M)$  be the set of  $n \times n$  matrices which avoid  $M$ . As we noted in Section 2.1, a permutation  $\sigma$  avoids another permutation  $\pi$  if and only if the permutation matrix corresponding to  $\sigma$  avoids the permutation matrix corresponding to  $\pi$ . So, given  $\pi$ ,

let  $P$  be the permutation matrix of  $\pi$ . So if  $T_n(P)$  contains all  $n \times n$  (0,1)-matrices that avoid  $P$ , it contains (a fortiori) all of the  $n \times n$  permutation matrices that avoid  $P$ . In particular we have  $|T_n(P)| \geq S_n(\pi)$ .

Assuming the Füredi–Hajnal conjecture, Klazar proved the following statement in [35], which in turn implies Conjecture 2.1.1:

**Theorem 2.2.6.** *For any permutation matrix  $P$  there exists a constant  $c = c_P$  such that  $|T_n(P)| \leq c^n$ .*

*Proof.* Using  $f(n, P) = O(n)$  the statement of the theorem follows from the following simple recursion:

$$|T_{2n}(P)| \leq |T_n(P)| 15^{f(n, P)}.$$

To prove the recursion we map  $T_{2n}(P)$  to  $T_n(P)$  by partitioning any matrix  $A \in T_{2n}(P)$  into  $2 \times 2$  blocks and replacing each all-zero block by a 0-entry and all other blocks by 1-entries. As we saw in Lemma 2.2.1 the resulting  $n \times n$  matrix  $B$  avoids  $P$ . Any matrix  $B \in T_n(P)$  is the image of at most  $15^w$  matrices of  $T_{2n}(P)$  under this mapping where  $w$  is the number of 1-entries in  $B$ . Here  $w \leq f(n, P)$  so the recursion and the theorem follow.  $\square$

The reduction also provides a nice characterization in the theory of excluded matrices:

**Corollary 2.2.7.** *For any (0,1)-matrix  $P$ , we have  $\log(|T_n(P)|) = O(n)$  if and only if  $P$  has at most a single 1-entry in each row and column.*

*Proof.* The matrices in the characterization are the submatrices of permutation matrices. For these matrices  $\log(|T_n(P)|) = O(n)$  follows from Theorem 2.2.6. For other matrices  $P$ ,  $T_n(P)$  contains all of the  $n \times n$  permutation matrices (a total of  $n!$ ), so  $\log(|T_n(P)|) = \Omega(n \log n)$ .  $\square$

### 2.2.3 Proof of the Alon–Friedgut Conjecture

Here, we reproduce another result of Klazar, originally from [35], which directly implies Conjecture 2.1.2.

**Theorem 2.2.8.** *Let  $c(k)$  be the constant in Theorem 2.2.5 and let  $a$  be word constructed from a permutation  $\sigma$  as in Conjecture 2.1.2. Then*

$$l_k(a, n)2 \leq k^2 \binom{c(k)}{k} n.$$

*Proof.* Let  $b$  be a word of size  $l_k(a, n)$  that avoids  $a$ . We build a  $(0,1)$ -matrix  $B$  as follows: we break  $b$  into  $n$  (contiguous) blocks, each of size  $q(n) = \lfloor l_k(a, n)/n \rfloor$ , and we set  $B_{i,j} = 1$  if and only if the  $j^{\text{th}}$  block contains  $i$  as an entry. Let  $\alpha$  be the least number of 1-entries to appear in a column of  $B$ .

**Claim 1:**  $\alpha \leq c(k)$

*Proof.* Assume not. Then there are more than  $c(k) \cdot n$  1-entries in  $A$ , and so by Theorem 2.2.5,  $A$  contains any permutation matrix of size  $k$ , and in particular, it contains  $\sigma$ , and so  $a$  can be found in  $b$  (a contradiction).  $\square$

**Claim 2:**  $q(n) \leq k^2 \cdot \binom{\alpha}{k}$

*Proof.* Consider the block that generated the column with  $\alpha$  1-entries. We break that block up into *miniblocks* size  $k$  (note that there are no repeated elements inside a miniblock, due to  $k$ -sparseness condition). If there are more than  $k \binom{\alpha}{k}$  miniblocks, then the same miniblock appears  $k$  times. These  $k$  miniblocks would then contain every permutation (and, in particular,  $\sigma$ ), so it must be that  $q(n) \leq k^2 \binom{\alpha}{k}$ .  $\square$

Putting the claims together, we have that

$$|b| \leq 2q(n) \cdot n \leq 2k^2 \cdot \binom{\alpha}{k} \cdot n \leq 2k^2 \cdot \binom{c(k)}{k} \cdot n$$

as required.  $\square$

#### 2.2.4 Graph Exclusion

We first remark that the  $(0,1)$ -matrix containment defined in Section 2.1.1 is equivalent to ordered containment in ordered bipartite graphs. The easiest way to see this is to consider

the bijection between an ordered bipartite graph  $G = (U \cup V, E)$  and its bipartite adjacency matrix  $A$ , defined as  $A_{i,j} = 1$  if and only if  $\{u_i, v_j\} \in E$  for all  $u_i \in U$  and  $v_j \in V$ .

For a graph  $K = ([k], F)$ , we define  $\text{gex}_{<}(n, G')$  to be the maximum number  $|E|$  of edges in a graph  $G = ([n], E)$  that does not contain  $K$  as an ordered subgraph. We represent a permutation  $\pi = a_1 a_2 \dots a_k$  of  $[k]$  by the graph

$$P(\pi) = ([2k], \{\{i, k + a_i\} : i \in [k]\}).$$

We will show the following bound as a direct corollary to Theorem 2.2.5:

**Corollary 2.2.9.** *Let  $\pi \in S_k$ . Then  $\text{gex}_{<}(n, P(\pi)) = O(n)$ .*

*Proof.* Let  $G = ([n], E)$  be a graph that avoids  $P(\pi)$ , and let  $A$  be the modified adjacency matrix  $A_{ij} = 1$  if and only if  $\{i, j\} \in E$  and such that  $i < j$ . Given  $\pi \in S_k$ , form  $\sigma \in S_{k+1}$  as  $\sigma(1) = k + 1, \sigma(i) = \pi(i - 1)$  for  $i = 2, \dots, k + 1$ , and let  $P_\sigma$  be the permutation matrix corresponding to  $\sigma$ . It is easy to see that if  $A$  contains the matrix  $P_\sigma$ , then  $G$  contains  $P(\pi)$ , and it is also easy to see that the number of edges in  $G$  is the number of 1-entries in  $A$ . Hence by Theorem 2.2.5, the number of 1-entries in  $A$ , and therefore the number of edges in  $G$ , is at most  $O(n)$ .  $\square$

## 2.3 Enumeration Bounds

The notion of “containment” studied in the previous section can be extended to more general structures. The following definition of hypergraph containment, first introduced by Klazar in [34], provides a general framework to investigate extremal problems for ordered structures (see [53] for recent revival and progress in this area).

### 2.3.1 Preliminaries

We say that a hypergraph  $\mathcal{H} = ([n], \mathcal{E})$  *contains* the hypergraph  $\mathcal{K} = ([m], \mathcal{F})$ , written  $\mathcal{G} \prec \mathcal{H}$ , if there exists an order preserving function  $f : [m] \rightarrow [n]$  and an injection  $g : \mathcal{F} \rightarrow \mathcal{E}$  such that  $f(F) \subset g(F)$  for every  $F \in \mathcal{F}$ . If no such  $f$  exists, we say that  $\mathcal{H}$  *avoids*  $\mathcal{K}$ . In essence,  $\mathcal{K} \prec \mathcal{H}$  means that  $\mathcal{K}$  can be obtained from  $\mathcal{H}$  by deleting edges, deleting vertices, and shrinking edges. Reordering the vertices, however, is not allowed. Note that a simple hypergraph can contain a non-simple hypergraph.



For a hypergraph  $\mathcal{F}$ , we associate the two functions  $\text{ex}_e(\cdot, \mathcal{F}) : \mathbb{N} \rightarrow \mathbb{N}$  and  $\text{ex}_i(\cdot, \mathcal{F}) : \mathbb{N} \rightarrow \mathbb{N}$ , where

$$\text{ex}_e(n, \mathcal{F}) = \max\{e(\mathcal{H}) : \mathcal{H} \not\preceq \mathcal{F}, \mathcal{H} \text{ is simple, and } v(\mathcal{H}) \leq n\}$$

$$\text{ex}_i(n, \mathcal{F}) = \max\{i(\mathcal{H}) : \mathcal{H} \not\preceq \mathcal{F}, \mathcal{H} \text{ is simple, and } v(\mathcal{H}) \leq n\}.$$

### 2.3.2 History

We first remark that this containment generalizes the (0,1)-matrix containment defined in Section 2.1.1. As mentioned in Section 2.2.4, the theory of ordered graph inclusion extends the theory of ordered (0,1)-matrix inclusion and so the theory of ordered hypergraph inclusion extends both. We will see that this containment also generalizes other well-studied situations in combinatorics, including the notion of noncrossing structures and Davenport–Schinzel sequences.

A *noncrossing hypergraph* is an (ordered) hypergraph having no four vertices  $a < b < c < d$  such that  $a, c$  lie in one edge  $A$  and  $b, d$  lie in another edge  $B$ , where  $A \neq B$ . Hence a hypergraph  $\mathcal{H}$  is noncrossing if and only if  $\mathcal{H} \not\preceq (\{1, 3\}, \{2, 4\})$ . In [24], Flajolet and Noy give a history of noncrossing configurations, dating the study of noncrossing graphs as far as the 1800’s. Noncrossing set partitions, especially, appear in many places in combinatorics and mathematics (see [44, 50]). Noncrossing hypergraphs serve as a common generalization of these two concepts, allowing edges of any size (as in partitions) as well as intersecting edges (as in graphs).

Let  $\lambda_d(n)$  be the size of the longest 2-sparse sequence  $a \in [n]^*$  such that  $a$  has no alternating subsequence of length greater than  $d$  (see Section 2.1.1 for definitions). Such a maximal sequence  $a$  is called a *Davenport–Schinzel sequence*. The study of  $\lambda_d(n)$  dated back to the original work of Davenport and Schinzel from the 1960’s [21]. It was a longstanding open problem whether  $\lambda_5(n)$  was linear in  $n$ , and it was not until the breakthrough paper [29] where Hart and Sharir showed that  $\lambda_5(n) = \Theta(n\alpha(n))$  (where  $\alpha(n)$  is the inverse Ackermann function). In this framework, we can characterize the Davenport–Schinzel sequences (for  $d = 5$ ) as exactly those set partitions  $S$  satisfying  $S \not\preceq (\{1, 3, 5\}, \{2, 4\})$ .

Soon after the original Hart–Sharir proof, the values of  $\lambda_d(n)$  were established for all  $d$

[3]. Since then, however, the theory has been generalized to multiple dimensions (see [26]) and more general patterns (see [32, 54]).

Balogh, Bollobás and Morris [8] recently derived Theorem 2.3.3 (their Theorem 2) and Corollary 2.3.4 (their Theorem 1) independently. The proofs in [8] are self-contained, adapting the ideas in Section 2.2.1 directly to the more general structures (rather than appealing to the results in [36]). In this manner, they are able to prove stronger statements, which in turn imply Theorem 2.3.3 and Corollary 2.3.4.

### 2.3.3 General Hypergraph Exclusion

Let  $\mathcal{K} = ([m], \mathcal{F})$ . Rather than finding bounds for particular  $\mathcal{K}$ , we will try to relate the quantities  $\text{ex}_e(n, \mathcal{K})$  and  $\text{ex}_i(n, \mathcal{K})$  in order to find bounds for all  $\mathcal{K}$ .

Let  $A, B \subset \mathbb{N}$  such that  $a < b$  for every element  $a \in A$  and  $b \in B$ . We will call such sets *separated* and write  $A < B$ . The following lemma is due to Klazar, his Theorem 2.3 in [36].

**Lemma 2.3.1.** *For  $\mathcal{K} = ([m], \mathcal{F})$ , we have that*

$$\text{ex}_e(n, \mathcal{F}) \leq \text{ex}_i(n, \mathcal{F}).$$

*If, in addition,  $\mathcal{K}$  has a pair of separated edges, then we also have*

$$\text{ex}_i(n, F) \leq (2v(F) - 1)(e(F) - 1)\text{ex}_e(n, F)$$

*so, in particular, for every permutation  $\pi \in S_k$ ,*

$$\text{ex}_i(n, P(\pi)) \leq (4k - 1)(k - 1)\text{ex}_e(n, P(\pi)).$$

Thus a linear upper bound on  $\text{ex}_i(n, P(\pi))$  follows directly from a linear upper bound on  $\text{ex}_e(n, P(\pi))$ . Such a bound on  $\text{ex}_e(n, P(\pi))$  can be derived using the techniques in [36] along with the graph bound in Corollary 2.2.9. To explain the reduction we need the notion of the *blow-up* of a graph.

Given a graph  $G$ , we say that the graph  $G'$  is an *m-blow-up* of  $G$  if for every edge coloring of  $G'$  by colors from  $\mathbb{N}$  such that every color is used at most  $m$  times, there exists a subgraph of  $G'$  that is order-isomorphic to  $G$  and that has no two edges with the same

color. For simplicity, the only blow-up that will be used in this paper will be the graph  $B(G)$  formed by replacing every vertex  $v \in V$  with an independent set  $I_v$  of size  $\binom{k}{2}$  and by placing all of the edges between  $I_v$  and  $I_u$  if and only if  $\{u, v\} \in E$ .

Theorem 3.1 in [36] gives the following bound, relating the extremal functions of a graph and the blow-up of the form mentioned above (see Section 2.2.4 for the relevant definitions):

**Lemma 2.3.2.** *Let  $G = (V, E)$  be a graph with  $|V| = k$ , and let  $B$  be its  $\binom{k}{2}$ -blow-up. Now assume that there exists a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\text{gex}_{<}(n, B) < n \cdot f(n)$  for all  $n$ . Then*

$$\text{ex}_e(n, G) < e(G) \cdot \text{gex}_{<}(n, G) \cdot \text{ex}_e(2f(n) + 1, G)$$

for all  $n$ .

Combining the bound in Corollary 2.2.9 with those in Lemma 2.3.1 and Lemma 2.3.2, we obtain the following result:

**Theorem 2.3.3.** *For every permutation  $\pi$ ,*

$$\text{ex}_i(n, P(\pi)) = O(n).$$

*Proof.* For  $m \in \mathbb{N}$  and a  $k$ -permutation  $\pi$ , we construct the permutation graph  $P(\pi')$  from  $P(\pi)$  by replacing every vertex of  $P(\pi)$  with an interval of  $k(m-1) + 1$  vertices (so each  $v \in [2k]$  becomes the interval  $I_v = [(v-1)(km-k+1) + 1, v(km-k+1)]$ ). Now for each edge  $\{u, v\}$  in  $P(\pi)$ , we place a perfect matching between the intervals  $I_u$  and  $I_v$ . Thus if we take any selection of one edge from each of the  $k$  perfect matchings, the resulting graph is order-isomorphic to  $P(\pi)$ . It should be noted that there are many such  $P(\pi')$ 's ( $\pi'$  is always a  $k^2(m-1) + k$ -permutation) but each of them is, by the pigeonhole principle, an  $m$ -blow-up of  $P(\pi)$ .

We set  $m = \binom{2k}{2}$ . By the graph bound in Corollary 2.2.9, there are constants  $c_\pi$  and  $c_{\pi'}$  such that

$$\text{gex}_{<}(n, P(\pi)) < c_\pi n \quad \text{and} \quad \text{gex}_{<}(n, P(\pi')) < c_{\pi'} n$$

for every  $n$ . We set  $B = P(\pi')$  and  $f(n) = c_{\pi'}$  and apply the bound in Lemma 2.3.2 to get the linear bound

$$\text{ex}_e(n, P(\pi)) < kc_\pi \cdot \text{ex}_e(2c_{\pi'} + 1, P(\pi)) \cdot n.$$

Finally, by applying the bound in Lemma 2.3.1, we get that

$$\text{ex}_i(n, P(\pi)) < k(k-1)(4k-1)c_\pi \cdot \text{ex}_e(2c_{\pi'} + 1, P(\pi)) \cdot n,$$

proving the claim.  $\square$

Klazar posed the following six extremal and enumerative conjectures in [34]:

C1: The number of simple  $\mathcal{H}$  such that  $v(\mathcal{H}) = n$  and  $\mathcal{H} \not\prec P(\pi)$  is at most  $c_1^n$ .

C2: The number of maximal simple  $\mathcal{H}$  with  $v(\mathcal{H}) = n$  and  $\mathcal{H} \not\prec P(\pi)$  is at most  $c_2^n$ .

C3: For every simple  $\mathcal{H}$  with  $v(\mathcal{H}) = n$  and  $\mathcal{H} \not\prec P(\pi)$ , we have  $e(\mathcal{H}) < c_3 n$ .

C4: For every simple  $\mathcal{H}$  with  $v(\mathcal{H}) = n$  and  $\mathcal{H} \not\prec P(\pi)$ , we have  $i(\mathcal{H}) < c_4 n$ .

C5: The number of simple  $\mathcal{H}$  with  $i(\mathcal{H}) = n$  and  $\mathcal{H} \not\prec P(\pi)$  is at most  $c_5^n$ .

C6: The number of  $\mathcal{H}$  with  $i(\mathcal{H}) = n$  and  $\mathcal{H} \not\prec P(\pi)$  is at most  $c_6^n$ .

He showed that all six conjectures hold for a large class of permutations  $\pi$  and that they hold for every  $\pi$  in weaker forms: with almost linear and almost exponential bounds (respectively). Conjecture C4, however, is precisely the statement of Theorem 2.3.3, and it is easy to extend the proof given in this paper to affirm that all six conjectures hold for every permutation  $\pi$ .

We shall show how to amend the proofs in [34] to prove C1, and then note that C1 implies C2, C3, C5 and C6 via Lemma 2.1 of [34].

**Corollary 2.3.4.** *For every permutation  $\pi$  there exists a constant  $c_1 > 1$  (depending on  $\pi$ ) so that the number of simple hypergraphs on vertex set  $[n]$  avoiding  $P(\pi)$  is less than  $c_1^n$ .*

*Proof.* Theorems 2.4 and 2.5 in [34] show that the number of hypergraphs with a given weight  $i(H)$  that avoid  $P(\pi)$  is at most  $9^{(3^{2k}+2k)i(H)}$ . By Theorem 2.3.3, we are done.  $\square$

In view of the reformulation from permutations to bipartite graphs mentioned in Section 2.3.2, it is easy to see that Corollary 2.3.4 is an extension of the Stanley–Wilf conjecture (Conjecture 2.1.1). A related extension was proposed by Brändén and Mansour in Section 5

of [14]: they conjectured that the number of *sequences* over the ordered alphabet  $[n]$  which have length  $n$  and avoid  $\pi$  is at most exponential in  $n$ . These words can be represented by simple graphs  $G$  on  $[2n]$  in which every edge connects  $[n]$  and  $[n+1, 2n]$  and every  $x \in [n]$  has degree exactly 1; the containment of ordered words is then just the ordered subgraph relation, and so this extension is also subsumed in Corollary 2.3.4.

Corollary 2.3.4 subsumes yet another extension of the Stanley–Wilf conjecture proposed by Klazar [33], this time to set partitions. This extension is related to *k-noncrossing* and *k-nonnesting* set partitions whose exact enumeration was recently investigated by Chen et al. [17] and Bousquet-Mélou and Xin [13]. Consider, for a set partition  $S$  of  $[n]$ , the graph  $G(S) = ([n], E)$  in which an edge connects two neighboring elements of a block (not separated by another element of the same block). Then  $S$  is represented by increasing paths which are spanned by the blocks.  $S$  is a *k-noncrossing* (resp. *k-nonnesting*) partition if and only if  $P(12 \dots k)$  (resp.  $P(k(k-1) \dots 1)$ ) is not an ordered subgraph of  $G(H)$ . Thus Corollary 2.3.4 provides an exponential bound: for fixed  $k$ , the numbers of *k-noncrossing* and *k-nonnesting* partitions of  $[n]$  grow at most exponentially in  $n$ .

## 2.4 An Extension to *d-dimensional Matrices*

We now generalize the original Füredi–Hajnal conjecture from ordinary (0,1)-matrices to *d-dimensional* (0,1)-matrices. These can be viewed as *d*-ary relations (or, in hypergraph terminology, *d-uniform*, *d-partite* hypergraphs). We keep the matrix terminology, however, both for the sake of consistency and to highlight the similarities between the methods in this section and those in Section 2.2.1.

### 2.4.1 Preliminaries

We will call a  $(d+1)$ -tuple  $\mathcal{M} = (M; n_1, \dots, n_d)$  with  $M \subset [n_1] \times \dots \times [n_d]$  a *d-dimensional (0,1)-matrix*, and will refer to the elements of  $M$  as *1-entries* of  $M$ . We define the *size* of  $\mathcal{M}$  (written  $|\mathcal{M}|$ ) to be the cardinality of the set  $M$  (the number of 1-entries).

If  $\mathcal{F} = (F; k_1, \dots, k_d)$  and  $\mathcal{M} = (M; n_1, \dots, n_d)$  are two *d-dimensional* matrices, we say that  $F$  is *contained* in  $\mathcal{M}$ , written  $\mathcal{F} \prec \mathcal{M}$ , if there exist *d* order preserving functions  $f_i : [k_i] \rightarrow [n_i]$ ,  $i = 1, 2, \dots, d$ , such that for every  $(x_1, \dots, x_d) \in F$  we have

$(f_1(x_1), \dots, f_d(x_d)) \in M$ ; otherwise we say that  $\mathcal{M}$  *avoids*  $\mathcal{F}$ . We set  $f(n, \mathcal{F}, d)$  to be the largest size  $|\mathcal{M}|$  of a  $d$ -dimensional matrix  $\mathcal{M} = (M; n, \dots, n)$  that avoids a fixed  $d$ -dimensional matrix  $F$ .

Note that in hypergraph terminology,  $\mathcal{M} = (M; n_1, \dots, n_d)$  is nothing more than a  $d$ -partite,  $d$ -uniform (ordered) hypergraph with the  $i^{\text{th}}$  partition having  $n_i$  vertices. Then  $M$  would be the collection of edges (where  $(x_1, x_2, \dots, x_d)$  would be a 1-entry in  $\mathcal{M}$  if and only if  $\{x_1, x_2, \dots, x_d\}$  was an edge in the hypergraph).

For  $i \in [d]$ , we define the  $i^{\text{th}}$  *projection map*  $\rho_i : [n_1] \times \dots \times [n_d] \rightarrow [n_i]$  as  $\rho_i(x_1, \dots, x_d) = x_i$ . For  $t \in [d]$ , we define the  $t$ -*remainder* of  $\mathcal{M} = (M; n_1, \dots, n_d)$  to be the  $(d-1)$ -dimensional matrix  $\mathcal{N} = (N; n'_1, \dots, n'_{d-1})$  where

$$n'_i = \begin{cases} n_i & \text{for } i < t \\ n_{i+1} & \text{for } i \geq t \end{cases}$$

and where the collection  $N$  is defined to be

$$N = \{(e_1, \dots, e_{d-1}) : (e_1, \dots, e_{t-1}, x, e_t, e_{t+1}, \dots, e_{d-1}) \in M \text{ for some } x \in [n_t]\}.$$

Again, translating into the hypergraph terminology, the image of  $M$  by the projection  $\rho_i$  is obtained by intersecting the edges (i.e. 1-entries) with the  $i^{\text{th}}$  part in the partition, while the intersections with the union of all parts except the  $t^{\text{th}}$  one gives the  $t$ -remainder of  $M$  (in both cases we disregard multiplicity of edges).

Let  $I_1 < I_2 < \dots < I_r$  be a partition of  $[n]$  into  $r$  intervals and  $\mathcal{M} = (M; n, \dots, n)$  a  $d$ -dimensional matrix. We define the *reduction* of  $\mathcal{M}$  (with respect to the intervals) to be the  $d$ -dimensional matrix  $\mathcal{N} = (N; r, \dots, r)$  given by  $(e_1, \dots, e_d) \in N$  if and only if  $M \cap (I_{e_1} \times \dots \times I_{e_d}) \neq \emptyset$  (we could define the reduction operation for a general  $d$ -dimensional matrix and with distinct and general partitions in each coordinate but we will not need such generality).

The reduction of  $\mathcal{M}$  with respect to a partition  $I$  is a direct extension of the concepts of blocks and reduction first discussed in Section 2.2.1. That is, the edge  $\{x_1, x_2, \dots, x_d\}$  exists in the reduction if and only if there is at least one edge in  $\mathcal{M}$  which contains a vertex

from *each* of the blocks  $B_{x_1}, B_{x_2}, \dots, B_{x_d}$ . Here, the blocks are the parts of the underlying vertex partition each cut into pieces by the partition  $I$ .

For  $d \geq 2$ , we say that  $\mathcal{P} = (P; k, \dots, k)$  is a *d-dimensional permutation of  $[k]$*  if for every  $i \in [d]$  and  $x \in [k]$  there is a single 1-entry  $e \in P$  with  $\rho_i(e) = x$ . In the degenerate case  $d = 1$ , we define the only 1-dimensional permutation to be  $\mathcal{P} = (P; k)$  with  $P = [k]$ . Note that there are exactly  $(k!)^{d-1}$  different  $d$ -dimensional permutations of  $[k]$ , and that each has exactly  $k$  1-entries. For example, the 2-dimensional permutations  $\mathcal{P} = (P; k, k)$  are precisely the  $k \times k$  permutation matrices, as defined in Section 2.1.1.

In hypergraph terminology, the  $d$ -dimensional permutations of  $[k]$  would be the set of (ordered) perfect matchings of the complete  $d$ -uniform,  $d$ -partite hypergraph on  $kd$  vertices.

#### 2.4.2 A Generalization of Füredi–Hajnal to $d$ Dimensions

Our goal is to prove the following claim, which is a generalization of Conjecture 2.1.3:

**Conjecture 2.4.1.** *For every fixed  $d$ -dimensional permutation  $\mathcal{P}$ ,*

$$f(n, \mathcal{P}, d) = O(n^{d-1}).$$

It is clear that for a  $d$ -dimensional permutation  $\mathcal{P}$  with  $|\mathcal{P}| > 1$  we have  $f(n, \mathcal{P}, d) \geq n^{d-1}$ , since fixing one coordinate and placing 1-entries in all possible positions will avoid all  $d$ -dimensional permutations<sup>1</sup>. Thus, for a  $d$ -dimensional permutation  $\mathcal{P}$  with  $|\mathcal{P}| > 1$ , Conjecture 2.4.1 would imply that

$$f(n, \mathcal{P}, d) = \Theta(n^{d-1}).$$

To prove Conjecture 2.4.1, we will need to consider every  $d$ -dimensional permutation of  $k$  simultaneously. The astute reader might have noticed that, in fact, the proof of Theorem 2.2.5 gives an upper bound for avoiding every permutation matrix. The proofs in this section, however, fundamentally require that we consider all permutations, and so we address this explicitly, setting

$$f(n, k, d) = \max_{\mathcal{P}} f(n, \mathcal{P}, d)$$

---

<sup>1</sup> $f(n, \mathcal{P}, d) = 0$  if  $|\mathcal{P}| = 1$ , which can only occur when  $k = 1$ .

where  $\mathcal{P}$  runs through all  $d$ -dimensional permutations of  $[k]$ .

We will make use of two observations, which generalize those made in Section 2.2.1. The first lemma generalizes Lemma 2.2.1 (and has an identical proof):

**Lemma 2.4.2.** *If  $\mathcal{M} = (M; n, \dots, n)$  avoids a  $d$ -dimensional permutation, then so does any reduction of  $\mathcal{M}$ .*

We will divide the 1-entries into blocks, just as before, and then find a concept of *largeness* that extends the idea of wide and tall blocks (in  $d$  dimensions, we will need  $d$  such concepts). Then we will count blocks as before, except that the direct count that we were able to obtain in Lemma 2.2.2 will need to be replaced by an inductive bound. In particular, we make the following observation:

**Lemma 2.4.3.** *For  $d \geq 2$ , and for any  $t \in [d]$ , the  $t$ -remainder of a  $d$ -dimensional permutation of  $[k]$  is a  $(d - 1)$ -dimensional permutation of  $[k]$ . Furthermore, each 1-entry of the resulting  $t$ -remainder can be completed in a unique way to an edge of the original permutation (by adding back the  $t^{\text{th}}$  coordinate).*

We now show that a  $d$ -dimensional matrix of big enough size must contain every  $d$ -dimensional permutation of  $k$ . The general idea is similar to that of Section 2.2.1.

**Lemma 2.4.4.** *Let  $d \geq 2$ ,  $m, n_0 \in \mathbb{N}$ . Then*

$$f(mn_0, k, d) \leq (k - 1)^d \cdot f(n_0, k, d) + dn_0 m^d \binom{m}{k} \cdot f(n_0, k, d - 1).$$

*Proof.* Let  $\mathcal{M} = (M, mn_0, \dots, mn_0)$  be a  $d$ -dimensional matrix with  $f(mn_0, k, d)$  1-entries that avoids  $\mathcal{P}$ , a  $d$ -dimensional permutation of  $[k]$ . We aim to bound the size of  $\mathcal{M}$ .

We split  $[mn_0]$  into  $n_0$  intervals  $I_1 < I_2 < \dots < I_{n_0}$ , each of length  $m$ , and define, for  $i_1, \dots, i_d \in [n_0]$ ,

$$S(i_1, \dots, i_d) = \{e \in M : \rho_j(e) \in I_{i_j} \text{ for } j = 1, \dots, d\}.$$

Note that this partitions the set of 1-entries of  $\mathcal{M}$  into  $n_0^d$  (possibly empty) collections. We will call these collections *blocks* and we define a cover of the blocks by a total of  $dn_0 + 1$  sets  $\{U_0\} \cup \{U(t, j) : t \in [d], j \in [n_0]\}$  as follows:



- $U(t, j) = \{S(i_1, \dots, i_d) : i_t = j \text{ and } |\rho_t(S(i_1, \dots, i_d))| \geq k\};$
- $U_0 = \{ \text{blocks which are not in any } U(t, j) \}.$

Note that the total number of non-empty blocks is exactly the number of 1-entries in the reduction of  $\mathcal{M}$  with respect to the partition  $\{I_i\}$ . Since  $\mathcal{M}$  does not contain  $\mathcal{P}$ , the reduction of  $\mathcal{M}$  cannot contain  $\mathcal{P}$ , so the number of non-empty blocks is at most  $f(n_0, k, d)$ . Also note that any block  $B$  in  $U_0$  has at most  $(k-1)^d$  non-zero entries in it (because  $B \subseteq X_1 \times \dots \times X_d$  for some  $X_i \subset [mn_0]$  with  $|X_i| < k$ ). Hence

$$|\bigcup U_0| \leq (k-1)^d \cdot f(n_0, k, d).$$

Now fix  $t \in [d]$  and  $j \in [n_0]$ . Clearly,

$$|\bigcup U(t, j)| \leq m^d |U(t, j)|.$$

We assume, for contradiction, that  $|U(t, j)| > \binom{m}{k} \cdot f(n_0, k, d-1)$ . By the definition of  $U(t, j)$  and the pigeonhole principle, there are  $k$  numbers  $c_1 < c_2 < \dots < c_k$  in  $I_j$  and  $r$  blocks  $S_1, S_2, \dots, S_r$  in  $U(t, j)$  where  $r > f(n_0, k, d-1)$  such that for every  $S_a$  and every  $c_b$  there is an  $e \in S_a$  with  $\rho_t(e) = c_b$ . Let  $\mathcal{P}'$  be the  $t$ -remainder of  $\mathcal{P}$  and  $\mathcal{M}' = (M'; n_0, \dots, n_0)$  be the  $(d-1)$ -dimensional matrix arising from reducing  $(\bigcup_{i=1}^r S_i, n, \dots, n)$  with respect to the intervals  $I_i$  and then taking the  $t$ -remainder. Since  $|\mathcal{M}'| = r > f(n_0, k, d-1)$ ,  $\mathcal{M}'$  contains  $\mathcal{P}'$ . Thus among the blocks  $S_1, S_2, \dots, S_r$  there exist  $k$  of them — call them  $S_1, S_2, \dots, S_k$  — so that for any selection of  $k$  edges  $e_1 \in S_1, \dots, e_k \in S_k$  their  $t$ -remainders form a copy of  $\mathcal{P}'$ . Furthermore, due to the property of the blocks  $S_i$ , it is possible to select  $e_1, \dots, e_k$  so that their  $t$ -th coordinates agree with  $\mathcal{P}$ . Then  $e_1, \dots, e_k$  form a copy of  $\mathcal{P}$ , a contradiction. Therefore

$$|\bigcup U(t, j)| \leq m^d |U(t, j)| \leq m^d \binom{m}{k} \cdot f(n_0, k, d-1)$$

and

$$|\bigcup_{t,j} U(t, j)| \leq dn_0 m^d \binom{m}{k} \cdot f(n_0, k, d-1).$$

Combining this with the bound for  $U_0$  gives the stated bound.  $\square$

Theorem 2.4.1 is a direct consequence of the following lemma, which finds an explicit constant:

**Lemma 2.4.5.** *If  $m = \lceil k^{d/(d-1)} \rceil$ , then  $f(n, k, d) \leq k^d (dm \binom{m+1}{k})^{d-1} n^{d-1}$ .*

*Proof.* We will proceed by induction on  $d + n$ . For  $d = 1$  this holds since  $f(n, k, 1) = k - 1$  and, for  $n < k^2$ , this holds trivially. Now given  $n$  and  $d \geq 2$ , assume that the hypothesis is true for all  $d', n'$  such that  $d' + n' < d + n$ .

Let  $n_0 = \lfloor n/m \rfloor$  and

$$c_d = k^d \left( dm \binom{m+1}{k} \right)^{d-1}.$$

Using the inequality  $f(n, k, d) < f(mn_0, k, d) + dm n^{d-1}$ , Lemma 2.4.4, the inductive hypotheses, and  $n_0 \leq n/m$ , we get

$$f(n, k, d) < \left( \frac{(k-1)^d}{m^{d-1}} c_d + dm \left( \binom{m}{k} c_{d-1} + 1 \right) \right) n^{d-1}.$$

Since  $\frac{(k-1)^d}{m^{d-1}} \leq (1 - \frac{1}{k})^d \leq 1 - \frac{1}{k}$  and  $\binom{m}{k} c_{d-1} + 1 \leq \binom{m+1}{k} c_{d-1}$ , it follows that  $f(n, k, d) < c_d n^{d-1}$  with the  $c_d$  defined above.  $\square$

## 2.5 Further Research

It was noted that we have made no effort to optimize any of the constants in these sections, and it would be interesting to see if any of the constants could be drastically reduced. As far as constants in the Füredi–Hajnal bound are concerned, the best current bounds are due to Cibulka [19], however there still remains much room for improvement. The constants in the Stanley–Wilf conjecture are widely studied, however only small cases are known exactly. See [12] for the most recent bounds.

Section 2.3.3 considers general ordered hypergraphs avoiding permutations and Section 2.4.2 considers  $t$ -uniform,  $t$ -partite hypergraphs avoiding  $d$ -dimensional permutations. This begs the question as to whether a common generalization of these two results can be found. The approach used in [8] seems to be more adaptable to hypergraph extensions than the methods in this chapter, but there was no obvious way to adapt them for our purposes.

Lastly, it would be interesting to find an extension of the Stanley–Wilf conjecture to  $d$ -dimensional permutations. Attempting to extend Lemma 2.2.6 gives (extremely) poor

bounds, as the number of  $d$ -dimensional matrices grows much faster than the number of  $d$ -dimensional permutations. In particular, we conjecture the following:

**Conjecture 2.5.1.** *For all  $d, k \in \mathbb{N}$ , there exists a constant  $c = c_{k,d}$  such that the number of  $d$ -dimensional permutations of length  $n$  that avoid a fixed  $d$ -dimensional permutation of length  $k$  is  $O(c^n \cdot (n!)^{d-1})$ .*

Is it possible that there exists such a  $c$  independent of  $d$ ?

## CHAPTER III

### PATTERN AVOIDANCE IN CYCLIC ORDERINGS

In this chapter, we break from the theme of the previous chapter where we dealt with (products of) linear orders. Following the lead of Pinchasi and Radoičić [46], we will instead consider *cyclically ordered sequences* of distinct symbols from a finite alphabet. The goal is to apply our results to graphs drawn in the plane: the rotation sequences of vertices are cyclically ordered by their outgoing edges (see Section 3.2.1 for more specifics). We then apply these results to a number of different problems in geometric combinatorics.

#### 3.1 Cyclically Ordered Sequences

We begin by introducing the structures that we will be considering:

##### 3.1.1 Preliminaries

An *alphabet*  $\aleph$  is a finite set of distinct elements, called *symbols*. We will use the term *sequence* to denote a linearly ordered list of distinct symbols and the term *cyclic sequence* to denote a cyclically ordered list of distinct symbols (we will use the convention of the order going clockwise, although our results are independent of this choice). Note that sequences and cyclic sequences are quite similar, the only difference being that a cyclic sequence wraps around so that the last element precedes the first element. For example, the permutation  $(1, 2, \dots, 12)$  would be a linear ordering, whereas the location of these numbers on an analog clock would be a cyclic ordering of the same elements.

For a sequence or cyclic sequence  $A$  we write  $\overline{A}$  for the set of symbols in  $A$ . As in the previous sections, we will be concerned with subsets of  $\overline{A}$  endowed with the same ordering as they appeared in  $A$ . Thus a *subsequence* (likewise, *cyclic subsequence*) of  $A$  will be a (cyclic) sequence  $B$  such that  $\overline{B} \subseteq \overline{A}$  and such that the symbols in  $B$  have the same (cyclic) order as they do in  $A$ . In addition, we will create sequences from cyclic sequences by cutting them into *blocks*, which inherit a linear ordering from the underlying cyclic order (see Figure 1).

We shall say that two cyclic sequences are *intersection reverse* if the common elements appear in reversed cyclic order in the two cyclic sequences. A collection of cyclically ordered sequences  $s_1, s_2, \dots, s_m$  will be called *pairwise intersection reverse* if  $s_i$  and  $s_j$  are intersection reverse for all  $1 \leq i < j \leq m$ . We define intersection reverse for sequences just as for cyclic sequences: we say that the sequences  $A$  and  $B$  are *intersection reverse* if they induce inverse linear orders on  $\overline{A} \cap \overline{B}$ . If two sequences are not intersection reverse, we call them *singular*.

Note that if two sequences  $A$  and  $B$  have  $|\overline{A} \cap \overline{B}| \leq 1$ , then the sequences are trivially intersection reverse. The same holds for cyclic sequences  $A, B$  if  $|\overline{A} \cap \overline{B}| \leq 2$ .

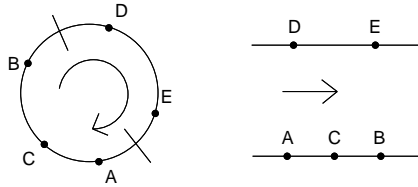
### 3.1.2 Previous Bounds

The goal of this chapter is to develop a bound on the total complexity of a collection of cyclically ordered sequences under the restriction that they be intersection reverse.

To gain some intuition, we will begin by proving the following simple bound:

**Theorem 3.1.1.** *Let  $A^1, A^2, \dots, A^m$  be a collection of cyclically ordered sequences of symbols from an alphabet size  $n$ , such that  $\sum \overline{A^i} = s$ . Then  $s \leq O(nm^{2/3} + m)$  and  $s \leq O(n^{2/3}m + n)$ .*

*Proof.* Note that three fixed symbols can appear in at most two different cyclic orders. Thus, if there are three symbols which appear together in three different cyclic sequences, two must appear in the same order, and so the collection cannot be intersection reverse. Hence it suffices to show that breaking the given bounds on  $s$  forces three of the cyclic sequences to have a common intersection of size at least 3.



**Figure 1:** Linearly ordered blocks inherited from the cyclic order.

The solution to this problem is originally due to Kővári, Sós, and Turán [31], and now has a number of proofs (though here we present the original one). Consider a bipartite graph  $G$  with parts  $X, Y$  of size  $n$  and  $m$  (corresponding to the alphabet and the sequences, respectively). An edge  $\{u, v\}$  will appear in  $G$  if and only if sequence  $A_v$  contains symbol  $s_u$ . Due to the discussion above, the intersection reverse property prevents  $G$  from containing  $K_{3,3}$  as a subgraph.

Consider, for each  $x \in X$ , the triples of edges (claws) incident to  $x$ . We can count these by looking at each  $x$ , or we can count them by looking at the triple of vertices in  $Y$  each claw is incident to. However, any such triple in  $Y$  can only be incident to (at most) 2 claws, since a third would create a  $K_{3,3}$ . Hence we have, (the leftmost inequality due to convexity)

$$|X| \binom{|E|/|X|}{3} \leq \sum_{x \in X} \binom{\deg(x)}{3} \leq 2 \binom{|Y|}{3}$$

Solving gives one of the inequalities and then reversing the roles of  $X$  and  $Y$  provides the other. □

The proof method in Theorem 3.1.1 only uses a very weak consequence of the intersection reverse property. In the breakthrough paper [46], Pinchasi and Radoičić were able to show that, in the case  $m = n$ , one could improve the  $O(n^{5/3})$  bound above to  $O(n^{8/5})$ . While this is an improvement, it still remains distant from the known lower bound:

**Theorem 3.1.2.** *Let  $A^1, A^2, \dots, A^m$  be a collection of cyclically ordered sequences of symbols from an alphabet size  $n$ , such that  $\sum \overline{A^i} = s$ . Then, for the case when  $m = n$ ,  $s \geq \Omega(n^{3/2})$ .*

*Proof.* Recall the observation at the end of Section 3.1.1 that cyclic sequences  $A$  and  $B$  are trivially intersection reverse if  $|\overline{A} \cap \overline{B}| \leq 2$ . Thus, using the same reduction to a bipartite graph that appeared in the previous theorem, it suffices to show the existence of a bipartite graph with parts size  $n$ ,  $\Omega(n^{3/2})$  edges, and no  $K_{2,2}$  subgraph. Again, this is a classic result and a number of constructions are now known. One of the more aesthetic constructions is due to Erdős [23], who noticed that the line/point intersection graph of a finite projective plane gives exactly this bound. □

Again, one might think that this is far from the correct lower bound, since the proof method uses an unnecessarily strong restriction to guarantee the intersection reverse property. However, we will show that this is not the case. In particular, we find the following upper bound that (up to a log factor) matches the known lower bound. Furthermore, we maintain the more general context of keeping  $m$  and  $n$  independent, providing a much more versatile theorem. In particular, we show the following theorem:

**Theorem 3.1.3.** *Let  $A^1, A^2, \dots, A^m$  be a collection of cyclically ordered sequences, each containing  $d$  distinct symbols from an alphabet size  $n$ . If these lists are pairwise intersection reverse, then*

$$d = O\left(\sqrt{n} \log n + \frac{n}{\sqrt{m}}\right).$$

Please note that, unlike the previous results, the sequences being considered in Theorem 3.1.3 are forced to be the same size. This eliminates a number of issues in the proof and, as will be shown in Corollary 3.1.8, the loss occurred by this restriction is strictly contained in the hidden constant.

### 3.1.3 Intersection reverse sequences

In this section we prove our main technical result, Theorem 3.1.3. Much of the proof follows the argument that Pinchasi and Radoičić used in [46]. We start with an overview of their techniques and comment on similarities and differences with the present proof.

Pinchasi and Radoičić break the cyclically ordered lists into linearly ordered blocks. They consider pairs of blocks from separate lists and pairs of symbols contained in both blocks. They distinguish between *same pairs* and *different pairs* according to whether the two symbols appear in the same or in different linear orders. They observe that any pair of symbols that appears in many blocks must produce almost as many same pairs as different pairs. On the other hand the intersection reverse property forces two cyclically ordered lists—unless most of their intersection is concentrated into a single pair of blocks—to contribute many more different than same pairs. Exceptional pairs of cyclically ordered lists are treated separately with techniques from extremal graph theory. They optimize in their choice for the length of the blocks.

We follow almost the same path, but instead of optimizing for block length we consider many block lengths (an exponential sequence) simultaneously. For two intersection reverse lists, no block length yields significantly more same pairs than different pairs. On the other hand, we will show that at least one of the block lengths actually gives many more different pairs than same pairs. As a consequence we do not have to bound “exceptional pairs” of lists separately.

For a sequence  $B$  and symbols  $a \neq b$  we define

$$f(B, a, b) = \begin{cases} 0 & \text{if } a \notin \overline{B} \text{ or } b \notin \overline{B}, \\ 1 & \text{if } a \text{ precedes } b \text{ in } B, \\ -1 & \text{if } b \text{ precedes } a \text{ in } B. \end{cases}$$

For two sequences  $B$  and  $B'$  we let

$$f(B, B', a, b) = f(B, a, b)f(B', a, b).$$

Notice that  $f(B, B', a, b) = 1$  for same pairs and  $f(B, B', a, b) = -1$  for different pairs, and that  $\sum f(B, B', a, b)$  corresponds to the difference between the number of same pairs and different pairs.

The next lemma is originally stated in [46]. We will use the notation  $\sum_{a \neq b}$  (both here and later in this section) to denote a sum taken over all ordered<sup>1</sup> pairs of distinct symbols  $a$  and  $b$ .

**Lemma 3.1.4.** *Let the cyclic sequences  $A$  and  $A'$  consist of the (linearly ordered) blocks  $B_1, \dots, B_k$  and  $B'_1, \dots, B'_{k'}$ , respectively. If  $A$  and  $A'$  are intersection reverse, then at most one of the pairs  $B_i, B'_j$  is singular. For this singular pair we have*

$$\sum_{a \neq b} f(B_i, B'_j, a, b) \leq |\overline{B_i} \cap \overline{B'_j}|.$$

For all of the other (intersection reverse) pairs  $B_i, B'_j$  we have

$$\sum_{a \neq b} f(B_i, B'_j, a, b) = |\overline{B_i} \cap \overline{B'_j}| - |\overline{B_i} \cap \overline{B'_j}|^2.$$

---

<sup>1</sup>Note that this causes an (intentional) double counting.



*Proof.* Let  $B_i$  and  $B'_j$  be blocks with common symbols appearing in the order  $a_1, \dots, a_l$  in  $B_i$ . Due to the intersection reverse property of  $A$  and  $A'$ , they appear in the order  $a_x, a_{x-1}, \dots, a_1, a_l, a_{l-1}, \dots, a_{x+1}$  in  $B'_j$  for some  $1 \leq x \leq l$ . Note that  $B_i$  and  $B'_j$  are singular if and only if  $x < l$ , and it is easy to verify that this can happen for at most a single pair of blocks (again due to the intersection reverse property). For a singular pair, we have

$$\begin{aligned} \sum_{a \neq b} f(B_i, B'_j, a, b) &= [2x(l-x)] - [x(x-1) + (l-x)(l-x-1)] \\ &= l - (l-2x)^2 \leq l. \end{aligned}$$

For all intersection reverse pairs, however, all pairs of symbols  $a \neq b$  from the intersection  $\overline{B_i} \cap \overline{B'_j}$  contribute  $-1$  to the sum.  $\square$

For the rest of the section, assume that we have the collection of pairwise intersection reverse cyclic sequences  $A^1, \dots, A^m$  from the theorem (recall that each consists of a  $d$ -element subset of a set of  $n$  symbols). Also, let  $p^{ij} = |\overline{A^i} \cap \overline{A^j}|$ , and  $p = \sum_{i \neq j} p^{ij}$ . First we bound  $p$  based on the limited size of the alphabet. For simplicity we assume  $dm > 2n$  (otherwise Theorem 3.1.3 is immediate).

**Lemma 3.1.5.**  $p \geq \frac{d^2 m^2}{2n}$  and  $\sum_{i \neq j} (p^{ij})^2 \geq \frac{p^2}{m^2}$ .

*Proof.* Let  $d_a$  be the number of times the symbol  $a$  appears among the cyclic sequences  $A^i$ . Then  $a$  contributes  $d_a^2 - d_a$  to  $p$ , so we have  $p = \sum_a d_a^2 - \sum_a d_a$ , where the summation is over the  $n$  different symbols  $a$ . We also have  $\sum_a d_a = dm$  as it is the sum of the sizes of the sequences  $A^i$ . Applying the inequality between the quadratic and the arithmetic mean and using  $dm > 2n$  we obtain

$$p = \sum_a d_a^2 - \sum_a d_a \geq \frac{1}{n} \left( \sum_a d_a \right)^2 - \sum_a d_a = \frac{d^2 m^2}{n} - dm \geq \frac{d^2 m^2}{2n}.$$

The second inequality in the lemma is also due to the inequality between the quadratic and arithmetic means, as

$$\sum_{i \neq j} (p^{ij})^2 \geq \frac{1}{m^2 - m} \left( \sum_{i \neq j} p^{ij} \right)^2 > \frac{p^2}{m^2}.$$

$\square$

We now define a recursive procedure for defining blocks. To begin, we split each  $A^i$  into two almost equal size (differing in size by at most 1) consecutive blocks  $A_0^i$  and  $A_1^i$ . Recursively, for a 0–1 sequence  $s$  we split the block  $A_s^i$  into two almost equal halves  $A_{s0}^i$  and  $A_{s1}^i$ . The cyclic order of  $A^i$  linearly orders the elements in each of these blocks. Let  $k = \lceil \log d \rceil < \log n + 1$ . Clearly for any 0–1 sequence  $s$  of length  $k$ , we have  $|\overline{A_s^i}| \leq 1$ .

For  $1 \leq i \leq m$  and  $1 \leq j \leq m$  we let

$$S^{ij} = \sum_{l=1}^k w_l \sum_{\substack{a \neq b \\ |s|=|t|=l}} f(A_s^i, A_t^j, a, b),$$

where the outer summation is taken over lengths  $1 \leq l \leq k$  and the inner summation is taken over all pairs of symbols  $a \neq b$  and all 0–1 sequences  $s$  and  $t$  of size  $|s| = |t| = l$ . We consider the pair  $(a, b)$  to be ordered, thereby double counting each unordered pair. The weights  $w_l$  in the formula are positive and we set them later. Our goal is to contrast a lower bound on  $\sum_{i \neq j} S^{ij}$  (or rather on the partial sum for fixed symbols  $a \neq b$ ) with upper bounds on the individual  $S^{ij}$ . Again we consider the  $(i, j)$  pairs to be ordered, resulting in another double counting.

The lower bound is straightforward:

**Lemma 3.1.6.**  $\sum_{i \neq j} S^{ij} \geq -md^2 \sum_{l=1}^k \frac{w_l}{2^l}.$

*Proof.* Notice that for fixed  $a, b$ , and  $l$  we get a perfect square when summing over all  $i$  and  $j$ . In particular,

$$\sum_{i=1}^m \sum_{j=1}^m S^{ij} = \sum_{l=1}^k w_l \sum_{a \neq b} \left( \sum_{i=1}^m \sum_{|s|=l} f(A_s^i, a, b) \right)^2 \geq 0$$

We can bound the  $S^{ii}$  terms separately as they are merely a (weighted) counting of the number of pairs contained in each block. Since  $|\overline{A_s^i}| < d/2^{|s|} + 1$ , we have

$$\sum_{i \neq j} S^{ij} = \sum_{i=1}^m \sum_{j=1}^m S^{ij} - \sum_{i=1}^m S^{ii} \geq 0 - \sum_{i=1}^m \sum_{l=1}^k 2w_l \sum_{|s|=l} \binom{|\overline{A_s^i}|}{2} \geq -md^2 \sum_{l=1}^k \frac{w_l}{2^l}.$$

as required. □

The upper bound, however, requires more effort.

**Lemma 3.1.7.** *For  $i \neq j$  we have*

$$S^{ij} \leq p^{ij} \sum_{l=1}^k w_l - \frac{(p^{ij})^2}{4 \sum_{l=1}^k \frac{1}{w_l}}.$$

*Proof.* We fix the indices  $i \neq j$  and consider the following quantities:

- $r_{st} = |\overline{A_s^i} \cap \overline{A_t^j}|$  and
- $Q_{st} = \sum_{a \neq b} f(A_s^i, A_t^j, a, b)$

where  $s$  and  $t$  are 0–1 sequences of equal length.

For a fixed length  $1 \leq l \leq k$ , the blocks  $A_s^i$  with  $|s| = l$  form a subdivision of  $A^i$ , while the blocks  $A_t^j$  with  $|t| = l$  form a subdivision of  $A^j$ . By Lemma 3.1.4, there is at most one singular pair  $(A_s^i, A_t^j)$  for any fixed length  $|s| = |t| = l$ . For these singular pairs we have

$$Q_{st} \leq r_{st},$$

while for the intersection reverse ones we have

$$Q_{st} = r_{st} - r_{st}^2.$$

Recall that any pair of sequences of length at most 1 is intersection reverse, so we do not find any singular pairs when  $|s| = |t| = k$ .

For a 0–1 sequence  $s$  of length  $|s| > 1$  let  $s'$  denote the sequence obtained from  $s$  by deleting its last digit, hence the block  $A_{s'}^i$  contains the smaller block  $A_s^i$ . We call a pair  $(s, t)$  of equal length 0–1 sequences a *leader pair* if  $(A_s^i, A_t^j)$  is intersection reverse and either  $|s| = |t| = 1$  or the pair  $(A_{s'}^i, A_{t'}^j)$  is singular. Since  $(A_{s'}^i, A_{t'}^j)$  is singular for at most one pair  $(s', t')$  of a fixed length, it follows that there can be at most 4 leader pairs  $(s, t)$  at the next bigger length. Furthermore, any symbol  $a \in \overline{A^i} \cap \overline{A^j}$  appears in  $\overline{A_s^i} \cap \overline{A_t^j}$  for exactly one leader pair  $(s, t)$ : the longest intersection reverse pair of blocks containing them (recall that we only consider pairs of blocks with equal length subscripts). Thus we have  $\sum_{(s,t) \in L} r_{st} = p^{ij}$  for the set  $L$  of leader pairs.

We use  $Q_{st} = r_{st} - r_{st}^2$  for leader pairs  $(s, t)$  only. For all other pairs, intersection reverse

or singular, we use  $Q_{st} \leq r_{st}$ :

$$\begin{aligned}
S^{ij} &= \sum_{l=1}^k w_l \sum_{|s|=|t|=l} Q_{st} \\
&\leq \sum_{l=1}^k w_l \sum_{|s|=|t|=l} r_{st} - \sum_{(s,t) \in L} w_{|s|} r_{st}^2 \\
&= p^{ij} \sum_{l=1}^k w_l - \sum_{(s,t) \in L} w_{|s|} r_{st}^2
\end{aligned}$$

since  $\sum_{|s|=|t|=l} r_{st} = p^{ij}$  for any fixed  $l$ . The Cauchy-Schwarz inequality gives

$$\left( \sum_{(s,t) \in L} w_{|s|} r_{st}^2 \right) \left( \sum_{(s,t) \in L} \frac{1}{w_{|s|}} \right) \geq \left( \sum_{(s,t) \in L} r_{st} \right)^2 = (p^{ij})^2.$$

Here  $\sum_{(s,t) \in L} (1/w_{|s|}) \leq 4 \sum_{l=1}^k (1/w_l)$ , so

$$\sum_{(s,t) \in L} w_{|s|} r_{st}^2 \geq \frac{(p^{ij})^2}{4 \sum_{l=1}^k \frac{1}{w_l}}$$

and we conclude that

$$S^{ij} \leq p^{ij} \sum_{l=1}^k w_l - \frac{(p^{ij})^2}{4 \sum_{l=1}^k \frac{1}{w_l}}$$

as claimed. □

Comparing the two estimates in the previous lemmas gives the theorem.

*Proof of Theorem 3.1.3.* Using the previous lemmas, we obtain

$$-m d^2 \sum_{l=1}^k \frac{w_l}{2^l} \leq \sum_{i \neq j} S^{ij} \tag{Lemma 3.1.6}$$

$$\leq \sum_{i \neq j} p^{ij} \sum_{l=1}^k w_l - \frac{\sum_{i \neq j} (p^{ij})^2}{4 \sum_{l=1}^k \frac{1}{w_l}} \tag{Lemma 3.1.7}$$

$$\leq p \sum_{l=1}^k w_l - \frac{p^2}{4 m^2 \sum_{l=1}^k \frac{1}{w_l}}. \tag{Lemma 3.1.5}$$

This inequality implies that (at least) one of two situations must be true: either  $p \leq 8m^2(\sum_{l=1}^k w_l)(\sum_{l=1}^k (1/w_l))$  or  $p^2 \leq 8d^2m^3(\sum_{l=1}^k (w_l/2^l))(\sum_{l=1}^k (1/w_l))$ . By Lemma 3.1.5, we have that  $p \geq d^2m^2/(2n)$ , so either

$$d \leq 4\sqrt{n} \sqrt{\sum_{l=1}^k w_l} \sqrt{\sum_{l=1}^k \frac{1}{w_l}} \quad \text{or} \quad d \leq \frac{6n}{\sqrt{m}} \sqrt{\sum_{l=1}^k \frac{w_l}{2^l}} \sqrt{\sum_{l=1}^k \frac{1}{w_l}}.$$

We choose the weights  $w_l$  now. Equal weights ( $w_l = 1$ ) yield  $d = O(\sqrt{n} \log n + n\sqrt{\log n}/\sqrt{m})$ , but we can improve on this bound by choosing

$$w_l = \frac{1}{1 + \frac{k}{2^{l/2}}}.$$

In this case  $\sum_{l=1}^k w_l \leq k$ ,  $\sum_{l=1}^k (1/w_l) \leq 4k$ , and  $\sum_{l=1}^k (w_l/2^l) \leq 3/k$ . Thus we either have  $d \leq 8k\sqrt{n}$  or  $d \leq 21n/\sqrt{m}$  and the statement of the theorem follows.  $\square$

We now show the easy extension to cyclic sequences that are not of uniform size.

**Corollary 3.1.8.** *Let  $A^1, A^2, \dots, A^m$  be a collection of cyclically ordered sequences of symbols from an alphabet size  $n$ , such that  $\sum \overline{A^i} = s$ . If the cyclic sequences are pairwise intersection reverse, then  $s = O(m\sqrt{n} \log n + n\sqrt{m})$ .*

*Proof.* Let  $c$  be the hidden constant in the statement of Theorem 3.1.3 (our proof gives  $c = 21$ ) and define  $t_k = c\sqrt{n} \log n + 2^k cn/\sqrt{m}$  for positive integers  $k$  (setting  $t_0 = 0$ ). We define  $m_k$  to be the number of cyclic sequences whose lengths lie in the interval  $(t_k, t_{k+1}]$ . For  $k > 1$ , if we prune each of the  $m_k$  sequences to be exactly length  $t_k$  and apply the uniform result derived in the previous section, we get that  $m_k \leq m/4^k$  (note this is trivially true for  $k = 0$  as well). Thus we have that the sum of the lengths of the sequences is at most

$$\sum_{k=0}^{\infty} m_k t_{k+1} = cm\sqrt{n} \log n + c \left( \sum_{k=0}^{\infty} 2^{k+1} m_k \right) \frac{n}{\sqrt{m}} = O(m\sqrt{n} \log n + n\sqrt{m})$$

$\square$

### 3.2 Consequences

The most important consequence of Theorem 3.1.3 deals with collections of *pseudo-circles*: simple closed Jordan curves, any two of which intersect at most twice, with proper crossings at each intersection. The result readily generalizes to unbounded open curves such as *pseudo-parabolas*, the graphs of continuous real functions defined on the entire real line such that any two intersect at most twice and they properly cross at these intersections.

Tamaki and Tokuyama [51] were the first to consider the problem of *cutting* pseudo-parabolas into *pseudo-segments*, i.e., subdividing the original curves into segments such that

any two segments intersect at most once<sup>2</sup>. In [51], it was shown that  $n$  pseudo-parabolas can be cut into  $O(n^{5/3})$  pseudo-segments using the bound in Theorem 3.1.1. This was extended to  $x$ -monotone pseudo-circles by Aronov and Sharir [5] and by Agarwal et al. [2]. It was also improved for certain collections of curves that admit a three-parameter algebraic parameterization to  $n^{3/2} \log^{\alpha^{O(1)}(n)}(n)$ , where  $\alpha$  is the inverse Ackermann function.

Previously, the best bound on the number of cuts needed for arbitrary collections of pseudo-parabolas and  $x$ -monotone pseudo-circles was  $O(n^{8/5})$  [2], which uses the result of Pinchasi and Radoičić in [46]. With our improvement of the latter result, we can prove that  $n$  pseudo-parabolas can be cut into  $O(n^{3/2} \log n)$  pseudo-segments. This substantially improves the previous bounds for arbitrary collections and is still slightly better than results on families with algebraic parameterization; we reduce a factor which grows slightly faster than polylogarithmically to a single logarithmic factor. In doing so, we are able to simplify the results in [2, 46, 51], as well as generalize them to the cases when the pseudo-parabolas and pseudo-circles are not necessarily  $x$ -monotone.

We will discuss these results in detail in Section 3.2.2. First, however, we formalize the concepts of planar graph embeddings that will be crucial in the reduction, and then show the relation to Theorem 3.1.3.

### 3.2.1 Self-Crossing Cycles of Length 4

A *plane curve* is a continuous function  $\phi : [0, 1] \rightarrow \mathbb{R}^2$ . Such a curve  $\phi$  is called *simple* or *non-crossing* if  $\phi$  is an injection and is called *closed* if  $\phi(0) = \phi(1)$ . In this chapter, we will blur the distinction between the function  $\phi$  and its image  $\{\phi(x) : 0 \leq x \leq 1\}$ , since the latter is an easier concept to visualize.

Given a graph  $G = (V, E)$ , we can create a *graph drawing*  $D : G \rightarrow \mathbb{R}^2$  in the plane by mapping the vertices  $V$  to points in  $\mathbb{R}^2$  and mapping the edges  $e = \{u, v\} \in E$  to simple curves  $\phi_e(x)$  such that  $\phi_e(0) = u$  and  $\phi_e(1) = v$ . Again, we will abuse notation and refer to the vertices and their images (also, the curves and their images) interchangeably. A graph drawing is called a *topological embedding* if the following properties hold:

---

<sup>2</sup>Such a separation turns out to be quite useful since pseudo-segments are much easier to work with than pseudo-parabolas and pseudo-circles

1. No vertex intersects the interior of any edge  $\{\phi(x) : 0 < x < 1\}$
2. Any two edges intersect at most a finite number of times
3. No two edges are tangent (all intersections are proper crossings)

If, in addition, all of the curves are straight lines, the drawing is called a *geometric embedding*.

A graph, together with a fixed topological embedding, will be called a *topological graph* (and *geometric graph* similarly). If a topological graph  $G$  has the property that no two edges have intersecting interiors, we say that  $G$  is *planar*. Finally, given a topological graph  $G$ , we will say that a subgraph  $K$  is *self-crossing* if the subgraph  $K$  (under the same embedding as  $G$ ) is not planar.

For a vertex  $v$  of a topological graph  $G$  let  $L_G(v)$  be the list of its neighbors ordered cyclically counterclockwise according to the initial segment of the connecting edge. Pinchasi and Radoičić [46] noticed the following simple fact:

**Fact 1.** *If the lists  $L_G(u)$  and  $L_G(v)$  are not intersection reverse for two distinct vertices  $u$  and  $v$  of the topological graph  $G$ , then  $G$  contains a self-crossing cycle of length 4. Moreover,  $u$  and  $v$  are opposite vertices of a cycle of length 4 in  $G$  that has two edges crossing an odd number of times.*

For the proof one only needs to consider drawings of the complete bipartite graph  $K_{2,3}$  (see details in [46]). Pinchasi and Radoičić used Fact 1 to bound the number of edges of a topological graph not containing a self-crossing  $C_4$ . They showed that such a graph on  $n$  vertices has  $O(n^{8/5})$  edges. Following in their footsteps, we use the same property to improve their bound to  $O(n^{3/2} \log n)$ . This bound is tight apart from the logarithmic factor since, as discussed in Theorem 3.1.2, there exist (abstract) simple graphs on  $n$  vertices with  $\Omega(n^{3/2})$  edges containing no  $C_4$ -subgraph (note that  $C_4$  and  $K_{2,2}$  are the same graph).

In [46], Pinchasi and Radoičić give a method for translating bounds like Theorem 3.1.3 into a bound on the number of edges of a topological graph not containing a self-crossing  $C_4$ . However, we are able to simplify the proof using Corollary 3.1.8.

**Corollary 3.2.1.** *If an  $n$ -vertex topological graph does not contain a self-crossing  $C_4$  it has  $O(n^{3/2} \log n)$  edges. The same holds if every pair of edges in every  $C_4$  subgraph cross an even number of times.*

*Proof.* The statements are direct consequences of Corollary 3.1.8 (in the case  $m = n$ ) using Fact 1, since the sum of the sizes of the lists of neighbors is the sum of the degrees, i.e., twice the number of edges.  $\square$

### 3.2.2 Cutting Number

Given a curve  $\phi$  in the plane, a *cut* is an operation that replaces  $\phi$  with subcurves  $\phi_1$  and  $\phi_2$ . Formally, a cut of  $\phi$  at the point  $p$  is the locus of points in  $\phi \setminus B(p, \epsilon)$  where  $B(p, \epsilon)$  is the ball of radius  $\epsilon$  around  $p$ . The point  $p$  and value  $\epsilon > 0$  must be such that the ball avoids all other curves and the intersection with  $\phi$  is continuous. So in particular,  $p$  cannot be chosen to be a vertex or a point of intersection of multiple curves.

By the Jordan curve theorem, a simple closed curve (such as a pseudo-circle) splits the plane into two open regions. We will call the bounded region the *interior* of the pseudo-circle. Following [51] we define a *lens* to be the union of two segments from distinct pseudo-circles if they form a closed curve. The two segments constituting the lens are called the *sides* of the lens. A side of a lens is *positive* if the interior of the corresponding pseudo-circle contains the other side of the lens. A lens is classified as a *lens-face* if both sides are positive, a *moon-face* if it has a positive and a negative side, and an *inverse-face* if it contains two negative sides. We will also consider each pseudo-circle itself to be a (degenerate) lens. A collection of *non-overlapping lenses* is a set of lenses such that no segment of any pseudo-circle is contained in more than one lens. The different types of non-degenerate lenses are illustrated in Figure 2.

Notice that non-overlapping lenses may cross each other. Our main result in this section is the following corollary:

**Corollary 3.2.2.** *An arrangement of  $n$  pseudo-circles can be cut at  $O(n^{3/2} \log n)$  points such that the resulting curves form a system of pseudo-segments.*

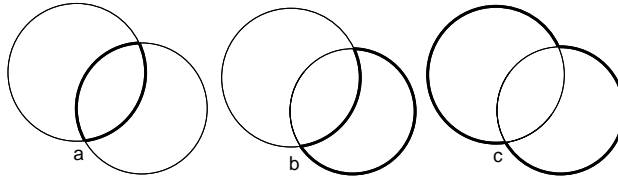


By combining the techniques presented in [2, 51] with Corollary 3.2.1, one can prove a version of Corollary 3.2.2 that is weaker in the sense that it is restricted to so called *x-monotone* pseudo-circles (defined in [2]). However, we give a simple and direct argument that does not require any additional monotonicity assumption on the pseudo-circles. Recall that this result slightly improves the best previous bound for (*x-monotone*) pseudo-circles with a three parameter algebraic representation as defined in [2] (such as ordinary circles) and substantially improves the previous bounds for pseudo-circles lacking such representation.

For a collection  $\mathcal{C}$  of pseudo-circles we let  $\nu(\mathcal{C})$  denote the maximum size of a non-overlapping family of lenses and  $\tau(\mathcal{C})$  denote the minimum number of cuts that transforms  $\mathcal{C}$  into a collection of pseudo-segments. The following lemma first appeared in [51], however, our proof takes a different approach. Apart from being shorter, it has the advantage of being extensible to collections of curves which are allowed to intersect more than twice.

**Lemma 3.2.3.**  $\tau(\mathcal{C}) = O(\nu(\mathcal{C}))$ .

*Proof.* We consider the lenses as a hypergraph: the vertices are the segments of the pseudo-circles connecting adjacent intersection points, the edges are the collections of these segments forming a lens. With this notation  $\nu(\mathcal{C})$  is the *packing (or matching) number* of this hypergraph, i.e., the maximum number of pairwise disjoint edges. Similarly,  $\tau(\mathcal{C})$  is the *transversal (or piercing) number* of the hypergraph, i.e., the minimum size of a collection of vertices that intersects every edge. After the cuts, the resulting curves will form a system of pseudo-segments if and only if we cut every lens at least once.



**Figure 2:** Examples of (a) a lens-face, (b) a moon-face, and (c) an inverse-face.

The relationship  $\tau(\mathcal{H}) \geq \nu(\mathcal{H})$  is true for any hypergraph  $\mathcal{H}$ , and much research has been focused on the connection between the packing and the transversal numbers. Tamaki and Tokuyama use a general result of Lovász [39] to compute their bound. We instead use the more specific result  $\tau = O(\nu)$  for the families of so called 2-intervals (a 2-interval is simply a union of two intervals of the real line). This was proved by Tardos [52], and later Kaiser [30] proved the tight bound  $\tau \leq 3\nu$ . Our lenses are almost 2-intervals: they consist of two intervals, but of pseudo-circles (not the real line).

We start by cutting every pseudo-circle at an arbitrary point. Now our (augmented) pseudo-circles can be mapped continuously to a set of disjoint intervals of the real line. For each lens that remains uncut, let the (disjoint) union of the images of the two segments forming the lens be a 2-interval. Using Kaiser's result we have  $\tau(\mathcal{C}) \leq 3\nu(\mathcal{C}) + n$ , where  $n$  is the number of pseudo-circles. Clearly  $n \leq \nu(\mathcal{C})$  as the collection of degenerate lenses is non-overlapping, so we have  $\tau(\mathcal{C}) \leq 4\nu(\mathcal{C})$  and this finishes the proof.  $\square$

Lemma 3.2.3 and the following lemma prove Corollary 3.2.2.

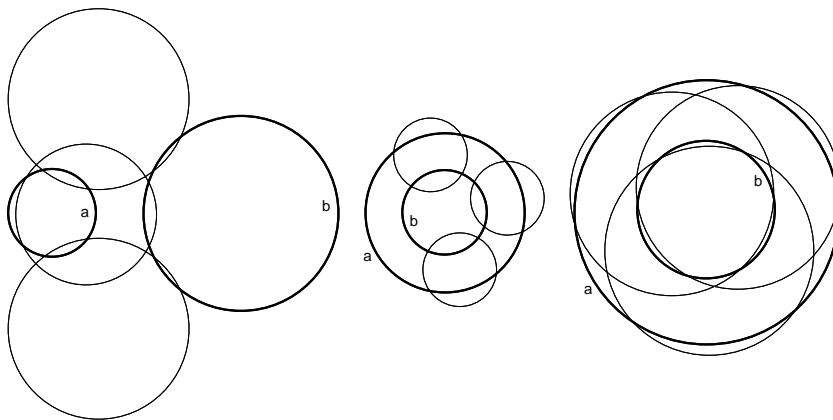
**Lemma 3.2.4.** *A collection of non-overlapping lenses in an arrangement of  $n$  pseudo-circles has  $O(n^{3/2} \log n)$  lenses.*

*Proof.* Given an arrangement  $\mathcal{C}$ , let  $L$  be a set of non-overlapping lenses with  $L^{lens}$ ,  $L^{moon}$ , and  $L^{inv}$  the sets of lens-faces, moon-faces, and inverse-faces in  $L$  (respectively). It is enough to prove the bound separately for each of these subsets, since the total number of degenerate lenses is only  $n$ .

For each  $c \in \mathcal{C}$ , and each subset  $L^k$  (for  $k = lens, moon, inv$ ) we make a list  $S_c^k$  consisting of all pseudo-circles  $c' \in \mathcal{C}$  that form a lens in  $L^k$  together with  $c$ . For the lenses in  $L^{moon}$ , however, we include  $c'$  in the list  $S_c^{moon}$  only if the corresponding lens has its positive side in  $c$  and its negative side in  $c'$  (otherwise it will appear in  $S_{c'}^{moon}$ ). We then order each of the lists  $S_c^k$  according to the counterclockwise cyclic order of these lenses around  $c$ . Since all of the lenses are non-overlapping, this cyclic order is well defined.

The main observation is that, for fixed  $k \in \{inv, lens, moon\}$ , the lists  $S_c^k$  must be pairwise intersection reverse. As in the proof of Fact 1, one can prove this observation

by considering the arrangements of 5 pseudo-circles forming six non-overlapping lenses. Notice that there are only a finite number of combinatorially different arrangements of 5 pseudo-circles in the plane. Instead of the simple but tedious case analysis we present three “counterexamples” where three pseudo-circles appear in the same cyclic order in the lists  $S_a$  and  $S_b$ . Here  $a$  and  $b$  are two of the pseudo-circles and we let  $S_a$  (respectively  $S_b$ ) be the cyclic list of all pseudo-circles that together with  $a$  (respectively with  $b$ ) form a lens in  $L$ . See Figure 3. Considering the lists  $S_a^{lens}$ ,  $S_a^{moon}$  and  $S_a^{inv}$  separately resolves the problem. In the first example, for the lists  $S_a$  we had to consider two lens-faces and a moon-face from  $L$ , while in the second example for  $S_a$  we considered moon-faces and for  $S_b$  we considered lens-faces. For the third example we considered only moon-faces in  $L$ , but the moon-faces considered for  $S_a$  have their negative (rather than positive) side on  $a$ .



**Figure 3:** Three “counterexamples” to the intersection reverse property of  $S_a$  and  $S_b$ .

By Corollary 3.1.8, the sum of the length of the lists  $S_c^k$  is  $O(n^{3/2} \log n)$  for each  $k$ . Hence the sum of the lengths of all of the lists is  $O(n^{3/2} \log n)$  as well—but this sum is at least the size of  $L$ .  $\square$

Corollary 3.2.2 naturally generalizes to collections of open Jordan curves including, for example, pseudo-parabolas. We call a collection of simple closed and open Jordan curves a *generalized pseudo-circle collection* if both ends of every open curve are at infinity, any two curves have at most two points of intersection, and the curves cross properly at each intersection.

**Corollary 3.2.5.** *A generalized pseudo-circle collection  $\mathcal{C}$  containing  $n$  curves can be cut at  $O(n^{3/2} \log n)$  points such that the resulting curve segments form a system of pseudo-segments.*

*Proof.* Given  $\mathcal{C}$ , we turn the arrangement into a system of  $n$  pseudo-circles and apply Corollary 3.2.2. Since there are a finite number of intersections, there is a sufficiently large circle  $D$  which contains all of them, together with all closed curves and all the segments of the open curves connecting two intersection points.

We modify the open curves in  $\mathcal{C}$  outside the circle  $D$  by closing them. We can choose the arcs closing up the open curves in such a way that any two of the curves intersect at most once outside  $D$ . Therefore any pair in the resulting family  $\mathcal{C}'$  intersects at most 3 times in total. Furthermore,  $\mathcal{C}'$  consists of closed curves with proper intersections, so any pair of them must cross an even number of times. Thus  $\mathcal{C}'$  is, in fact, a collection of pseudo-circles and Corollary 3.2.2 finishes the proof.  $\square$

### 3.2.3 Levels in Curve Arrangements

Corollary 3.2.2 also has many consequences in the study of *levels* in arrangements of curves. Tamaki and Tokuyama [51] were first to show the usefulness of cutting numbers in this area, and progress has since been made by Chan [15, 16].

Let  $\mathcal{C}$  be a finite collection of continuous real functions defined everywhere on the real line, such that any pair of curves intersects a finite number of times. We define the  $k^{th}$  level of  $\mathcal{C}$  to be the closure of the locus of points  $(x, y)$  on the curves in  $\mathcal{C}$  with  $|\{i : f_i(x) \leq y\}| = k$ . The  $k^{th}$  level consists of portions of the curves in  $\mathcal{C}$ , delimited by intersections between these curves. We will call the total number of curve segments in a level its *complexity*.

Chan [16] derives an upper bound on the complexity of a given level of a collection of pseudo-parabolas by recursively estimating the number of intersections that can appear within a range of levels by cutting into pseudo-segments. Our improved bound in Corollary 3.2.5 improves Chan's analysis. We sketch the reasoning below:

Let  $\mathcal{C}$  be a collection of  $n$  pseudo-parabolas and fix a level  $k$ . Let  $t_i$  stand for the number of intersections strictly between levels  $k - i$  and  $k + i$ . The main inequality (Lemma 3.1) in

[16] asserts that

$$t_i \leq 2i(t_{i+1} - t_i) + O(ni + \Lambda_i)$$

where  $\Lambda_i$  is the number of lenses (formed by the curves in  $\mathcal{C}$ ) lying strictly between levels  $k - i$  and  $k + i$ . Lemma 4.1 of the same paper gives the bound on  $\Lambda_i$ :

$$\Lambda_i = O(i^2 \nu(n/i))$$

where  $\nu(k)$  stands for the number of cuts needed to turn  $k$  pseudo-parabolas into a collection of pseudo-segments. By our Corollary 3.2.5 we have  $\nu(k) = O(k^{3/2} \log k)$ .

Putting these three inequalities together gives the recurrence

$$t_i \leq 2i(t_{i+1} - t_i) + O(i^{1/2} n^{3/2} \log n).$$

Using the fact that  $t_n = O(n^2)$  and solving the recurrence yields a bound on  $t_2$  and therefore on the complexity of the  $k^{th}$  level.

**Corollary 3.2.6.** *Let  $\mathcal{C}$  be a collection of  $n$  pseudo-parabolas. Then the maximum complexity of any level of  $\mathcal{C}$  is  $O(n^{3/2} \log^2 n)$ .*

The above corollary represents a substantial improvement over the previous bound of  $O(n^{8/5})$  for an arbitrary collection of pseudo-parabolas in [16]. For a collection possessing a three-parameter algebraic representation (as defined in [2]) the improvement is marginal, replacing a term which grows slightly faster than polylogarithmically with the term  $\log^2 n$ . These improvements carry over to levels of arrangements of algebraic curves of degree higher than two by the technique of *bootstrapping*, as developed in [16]. We do not state these slightly improved bounds here.

### 3.2.4 Incidences and Faces

Let  $\mathcal{C}$  be a set of curves and  $\mathcal{P}$  a set of points in the plane. We define  $I(\mathcal{C}, \mathcal{P})$  to be the number of *incidences* between  $\mathcal{C}$  and  $\mathcal{P}$ , that is, the number of pairs  $(c, p) \in \mathcal{C} \times \mathcal{P}$  such that curve  $c$  contains point  $p$ . We also define  $K(\mathcal{C}, \mathcal{P})$  to be the sum of the complexities of the faces in the arrangement  $\mathcal{C}$  which contain at least one point in  $\mathcal{P}$  (assuming now that they are not on the curves). Here a *face* is a connected component of the complement of

the union of the curves in  $\mathcal{C}$ , and the *complexity* of a face is defined to be the number of curve segments that comprise its boundary.

The results in [1, 5] relate the values of  $K(\mathcal{C}, \mathcal{P})$  and  $I(\mathcal{C}, \mathcal{P})$  (respectively) to the cutting numbers  $\tau(\mathcal{C})$  discussed above. The following bounds were shown:

**Lemma 3.2.7.** *If  $\mathcal{C}$  is a collection of  $n$  curves and  $\mathcal{P}$  is a set of  $m$  points, then*

$$I(\mathcal{C}, \mathcal{P}) = O(m^{2/3}n^{2/3} + m + \tau(\mathcal{C})),$$

$$K(\mathcal{C}, \mathcal{P}) = O(m^{2/3}n^{2/3} + m + \tau(\mathcal{C}) \log^2 n).$$

Thus, by Corollary 3.2.5, we have

**Corollary 3.2.8.** *If  $\mathcal{C}$  is a collection of  $n$  generalized pseudo-circles and  $\mathcal{P}$  is a set of  $m$  points, then*

1.  $I(\mathcal{C}, \mathcal{P}) = O(m^{2/3}n^{2/3} + m + n^{3/2} \log n)$
2.  $K(\mathcal{C}, \mathcal{P}) = O(m^{2/3}n^{2/3} + m + n^{3/2} \log^3 n)$

For curves that admit a three parameter algebraic representation (see [2]) Chan [16] is able to improve the incidence and complexity bounds in Corollary 3.2.8 by applying them separately to smaller subsets of the points and curves. Our results also improve these better bounds, but only marginally, and therefore we do not state them here.

### 3.3 Further Research

The results in this chapter raise a number of interesting questions. Corollary 3.2.1 is tight except possibly for the logarithmic factor as graphs with  $n$  vertices and  $\Omega(n^{3/2})$  edges are known which do not contain any  $C_4$  (see, for example, [23]). This also implies that the special cases of Theorem 3.1.3 and Corollary 3.1.8 when  $n = m$  are almost tight. Nevertheless, it would be interesting to know if the logarithmic factor is needed.

**Problem 3.3.1.** *Is the logarithmic factor needed in Corollary 3.2.1?*

Note that the statement of Corollary 3.2.1 is in regard to topological graphs in general. One may get a different answer for the restricted set of *geometric graphs*, that is, graphs with straight line segments as edges.

**Problem 3.3.2.** *Is the logarithmic factor needed in Corollary 3.2.1 if we consider geometric graphs rather than topological ones?*

The geometric consequences use Theorem 3.1.3 in the special case when  $n = m$ , but it is interesting to give bounds in the asymmetric cases as well. We define  $R(n, m)$  to be the maximum total length of  $m$  pairwise intersection reverse cyclic sequences over an alphabet of size  $n$ . With this notation Corollary 3.1.8 gives  $R(n, m) = O(m\sqrt{n} \log n + n\sqrt{m})$ . We collect here a few simple lower and upper bounds for  $R(n, m)$ .

A trivial consequence of the property that a collection of cyclic sequences are pairwise intersection reverse is that no three symbols appear together in three cyclic sequences. By Theorem 3.1.1, we have that  $R(n, m) = O(nm^{2/3} + m)$  and  $R(n, m) = O(n^{2/3}m + n)$ . The first bound supersedes the bound in Corollary 3.1.8 if  $m \geq n^{3/2}$ . The second bound supersedes the bound in Corollary 3.1.8 if  $m < n^{2/3}$ . So for these extremely large or small values of  $m$ , Corollary 3.1.8 is not tight.

The simplest constructions of intersection reverse cyclic sequences are constructions for collections of *subsets* intersecting each other in at most two elements. No matter how we order these subsets the resulting collection of cyclic sequences is pairwise intersection reverse. A simple construction for such subsets is any collection of circles in a finite plane. Taking all points of the plane and a subset of the circles gives  $R(n, m) = \Omega(m\sqrt{n})$  for  $m \leq n^{3/2}$ . Taking all circles and a subset of the points gives  $R(n, m) = \Omega(nm^{2/3})$  for  $m \geq n^{3/2}$ . A collection of singleton sets gives the trivial bound  $R(n, m) \geq m$ , which is better than the previous bounds for  $m > n^3$ . Pairwise disjoint sets provide the other trivial  $R(n, m) \geq n$  bound, which is better than the other bounds for  $m \leq \sqrt{n}$ .

The solid lines in the logarithmic scale diagram in Figure 4 shows the lower and upper bounds mentioned above. These bounds determine  $R(n, m)$  up to a constant factor for  $m \geq n^{3/2}$  and  $m \leq n^{1/3}$  and up to a logarithmic factor for  $n \leq m \leq n^{3/2}$ . In any construction proving better lower bounds than the ones above, a typical pair of cyclic sequences will need to intersect in many elements, so the cyclic order becomes essential in such a construction. We present such a construction below, proving  $R(n, m) = \Omega(n^{5/6}m^{1/2})$  for  $n^{1/3} < m < n^{2/3}$ . This bound is represented in Figure 4 by the dashed line. The area

of “uncertainty” is shaded. Even with this construction, the upper and lower bounds for  $R(n, m)$  are far apart for  $n^{1/3} < m < n$ .

**Construction 1.** The construction is based on a construction of Gy. Elekes [22] of a set of axis-aligned parabolas and a set of points with a large number of incidences. For integers  $b \geq a \geq 1$  consider the subset  $P = \{(i, j) : |i| \leq a, |j| \leq 3a^2b\}$  of the integer grid and consider the collection  $\mathcal{C}$  of parabolas (and lines) given by  $y = ux^2 + vx + w$  with integers  $u, v$ , and  $w$  satisfying  $|u| \leq b$ ,  $|v| \leq ab$  and  $|w| \leq a^2b$ . We have  $m = |P| = (2a + 1)(6a^2b + 1) = \Theta(a^3b)$  and  $n = |\mathcal{C}| = (2b + 1)(2ab + 1)(2a^2b + 1) = \Theta(a^3b^3)$ . Clearly, each curve in  $\mathcal{C}$  contains a point in  $P$  for each possible  $x$  coordinate, a total of  $2a + 1$  points. For each  $p \in P$  we define the linearly ordered list  $B_p$  of all the curves in  $\mathcal{C}$  passing through  $p$ . We order the list  $B_p$  according to the slopes of the curves at  $p$  (breaking ties arbitrarily). As a result we get  $m$  linearly ordered lists of subsets of the set of  $n$  symbols. Since axis-aligned parabolas form a collection of pseudo-parabolas – any pair intersects at most twice (and tangent parabolas have no further points in common) – it is easy to verify that these lists are intersection reverse. Their total length is the number of incidences between  $P$  and  $\mathcal{C}$ , which is  $\Theta(a^4b^3) = \Theta(n^{5/6}m^{1/2})$ .

**Problem 3.3.3.** *Is it possible to find  $n^{2/3}$  pairwise intersection reverse cyclic sequences over an alphabet of size  $n$  such that their total lengths sum to significantly more than  $n^{7/6}$ ?*

Note that for  $m = n^{2/3}$  both constructions give cyclic sequences with total size  $\Theta(n^{7/6})$ . One of the constructions is based on finite geometry, the other on Euclidean geometry. It seems to be hard to combine these constructions for a better result. The upper bound (provided both by Corollary 3.1.8 and Theorem 3.1.1) is  $O(n^{4/3})$ .

As Figure 4 shows, it is unclear as to whether the  $n\sqrt{m}$  term in Corollary 3.1.8 gives a tight bound for  $R(n, m)$  in any range. We claim that its appearance is meaningful, however. The total length of the sequences needs to be above this threshold in order for a typical pair of symbols to appear together in many cyclic sequences – a property which is necessary in our estimate that not many more different than same pairs exist. If a typical pair of symbols appears together in only two cyclic sequences, it is possible that



they only contribute different pairs. This happens in the above construction as well; since we construct linearly ordered (rather than cyclic) sequences that are pairwise intersection reverse, no “same pair” ever appears.

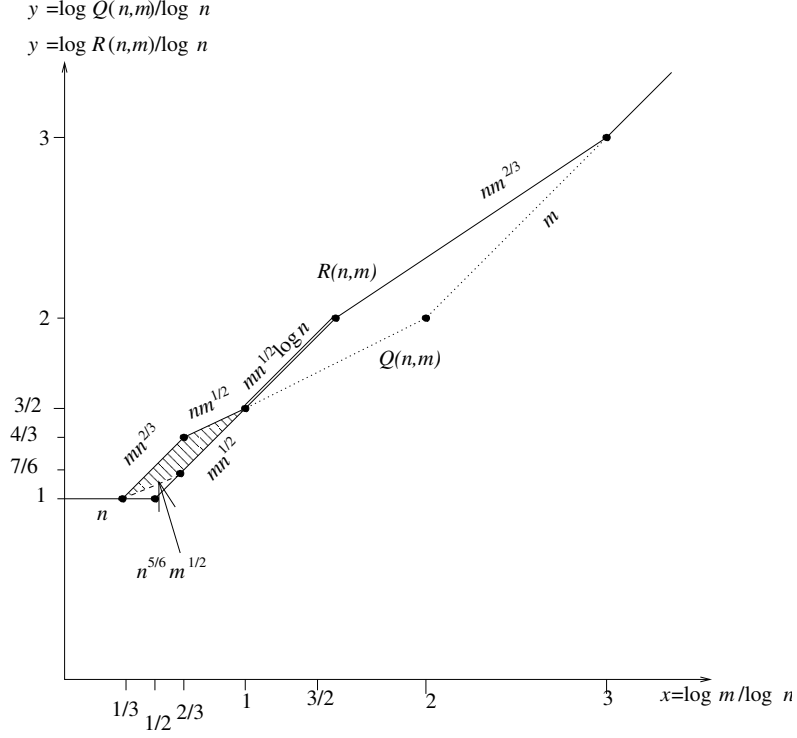
One can ask the same extremal question about linearly ordered sequences. Let  $Q(n, m)$  stand for the maximum total length of  $m$  pairwise intersection reverse sequences over an  $n$  element alphabet. In this case two symbols cannot appear together in three sequences. Theorem 3.1.1 therefore gives the bounds  $Q(n, m) = O(mn^{2/3} + n)$  and  $Q(m, n) = O(n\sqrt{m} + m)$ . For  $m \leq n/\log^2 n$  or  $m \geq n^3$  we get the same upper bounds that we did for  $R(n, m)$ . The upper bound for intermediate values of  $m$  is shown by the dotted line in Figure 4. One gets simple construction of intersection reverse sequences by considering set systems with pairwise intersection limited to singletons. Just as we noted in the case of cyclic sequences, this property ensures that the sequences are pairwise intersection reverse independent of the linear order chosen. The standard construction for such set systems is the set of lines in a finite plane, yielding  $Q(n, m) = \Omega(n\sqrt{m})$  for  $m \geq n$  and  $Q(n, m) = \Omega(m\sqrt{n})$  for  $m \leq n$ . The bounds  $Q(n, m) \geq n$  and  $Q(n, m) \geq m$  are trivial (just as before). These bounds determine  $Q(n, m)$  up to a constant factor for  $m \leq n^{1/3}$  and  $m \geq n$ . Notice that the construction using parabolas in the plane yield pairwise intersection reverse linearly ordered sequences and so we have  $Q(n, m) = \Omega(n^{5/6}m^{1/2})$  for  $n^{1/3} \leq m \leq n^{2/3}$ . Surprisingly, the “area of uncertainty” for  $Q(n, m)$  is exactly the same parallelogram as it is for  $R(n, m)$ . Only when  $n < m < n^3$  do the bounds for  $Q(n, m)$  and  $R(n, m)$  diverge. We do not know if allowing for cyclic sequences can yield longer intersection reverse collections in the case when  $m < n$ .

**Problem 3.3.4.** *Does  $R(n, m) = O(Q(n, m))$  hold for  $m < n$ ?*

As far as pseudo-circles are concerned, our result is conjectured to be far from optimal. The best known construction is a set of  $n$  pseudo-circles that needs  $\Omega(n^{4/3})$  cuts before it becomes a collection of pseudo-segments.

**Problem 3.3.5.** *What is the tight bound for the number of non-overlapping lenses in an arrangement of  $n$  pseudo-circles?*

As noted in Section 3.2, the results in this chapter generalize previous results in the



**Figure 4:** Bounds and area of uncertainty for  $R(n, m)$  and  $Q(n, m)$ .

respect that the curves no longer need to be  $x$ -monotone. However, there are certain extensions that can no longer be achieved. Chan [15] proved an *intersection-sensitive* bound, that is, a bound which is stated as a function of the total number of intersections. Previous papers [2, 15] are able to give such bounds for collections of  $x$ -monotone curves, but the methods break down when  $x$ -monotonicity is dropped.

**Problem 3.3.6.** Find an *intersection-sensitive* extension to Corollary 3.2.5.

## CHAPTER IV

### AN ENTROPY RESULT FOR SUMSETS

Given an abelian group  $(G, +)$  and sets  $A, B \subseteq G$ , the set  $A + B = \{a + b : a \in A, b \in B\}$  is called a *sumset*. The attempt to bound the cardinality of a sumset with respect to its components has become a popular area of analytic combinatorics. In this chapter, we employ an information theoretic method using *mathematical entropy* to develop a new collection of inequalities.

#### 4.1 Entropy

We first give a general review of entropy, as well as some of the properties that will be useful.

##### 4.1.1 Preliminaries

Let  $\Omega$  be a finite sample space,  $\mathcal{F}$  a  $\sigma$ -field on  $\Omega$ , and  $\mu$  a probability measure on  $\mathcal{F}$ , so that  $(\Omega, \mathcal{F}, \mu)$  is a discrete probability space. Now let  $X : \Omega \rightarrow \mathbb{R}$  be a random variable with  $Rng(X)$  denoting the range of  $X$ . Then the *entropy of  $X$*  is defined to be

$$H(X) = \sum_{x \in Rng(X)} \Pr(X = x) \log \left( \frac{1}{\Pr(X = x)} \right).$$

Note that, since the probability space is discrete, the sum only has a finite number of terms and therefore is well-defined. Given two random variables,  $X, Y$ , the *joint entropy* is defined to be

$$H(XY) = \sum_{x \in Rng(X)} \sum_{y \in Rng(Y)} \Pr(X = x, Y = y) \log \left( \frac{1}{\Pr(X = x, Y = y)} \right).$$

The *conditional entropy* is defined to be  $H(X|Y) = H(XY) - H(Y)$ .

Some important properties of entropy are the following (see [20], e. g., for proofs):

1.  $H(X) \leq \log |Rng(X)|$ , with equality if and only if  $X$  has the uniform distribution.

2.  $H(X|Y) \geq 0$ , with equality if and only if there exists a deterministic function  $g$  such that  $X = g(Y)$ .
3.  $H(XYZ) + H(X) \leq H(XY) + H(XZ)$ .

The last property mentioned implies that for sets of random variables  $A, B$ , one has  $H(A \cup B) + H(A \cap B) \leq H(A) + H(B)$ , a property known as *submodularity*.

Given a set  $S$ , let  $\hat{S} = \{T : T \subseteq S\}$  and for  $s \in S$ , let  $\hat{S}(s) = \{T : T \subseteq S, s \in T\}$ . A *fractional covering* of  $S$  is a function  $w : \hat{S} \rightarrow \mathbb{R}$  such that

$$\sum_{T \in \hat{S}(s)} w(T) \geq 1 \quad \text{for all } s \in S.$$

A function  $f : \hat{S} \rightarrow \mathbb{R}$  is called *fractionally subadditive* if, for all fractional covers  $w$  of  $S$ ,

$$f(S) \leq \sum_{T \subseteq S} w(T) f(T).$$

The following result is a well-known (some might say “folklore”) theorem from economics, where submodular functions are of particular interest (see [41] for a proof):

**Lemma 4.1.1.** *Let  $f$  be a submodular function on a ground set  $S$ . Then  $f$  is fractionally subadditive.*

#### 4.1.2 History

Information theoretic ideas have long been used for various combinatorial problems; however, the link has been much more evident as new approaches using entropy have been developed. Perhaps the first steps in this direction appeared in papers of Körner [38] and Pippenger [47], but one of the most crucial results was a black-box technique (now known as *Shearer’s Lemma*) for generating inequalities that exploited the powerful convexity properties of entropy [18]. Since then, there have been a number of generalizations and adaptations of Shearer’s Lemma to various situations (see, for example, [25, 41, 48]).

In particular, it has been noted recently [28] that a collection of inequalities known as *Han’s Inequalities* can be applied to sumset problems in much the same way that they can be applied to characteristic functions (the usual situation). In fact, we show that fractional

subadditivity (which contain Han's Inequalities and Shearer's Lemma as subcases) can be applied to a class of functions that contain both sumsets and characteristic functions as a subcase. The ideas extend an argument of [28] (more precisely, an idea in the proof of Theorem 1.2 in their paper), making further use of entropy, and showing general fractional subadditivity properties that imply some of the results and conjectures in [28] as easy corollaries.

## 4.2 Fractional Subadditivity of Deterministic Set Functions

The important property of sumsets that we wish to exploit is that, for a fixed element  $a$ , the sum  $a + b$  depends only on  $b$  (no further knowledge about how  $a$  and  $b$  relate is needed). This idea leads to a more general class of functions.

### 4.2.1 Deterministic Set Functions

Let  $X_1, X_2, \dots, X_k$  be finite sets. Any subset  $S \subseteq [k]$  corresponds to a different product space  $X_S = \prod_{i \in S} X_i$ . For sets  $S \subseteq T \subseteq [k]$ , we define the projection function  $\pi_S : X_T \rightarrow X_S$  in the natural way:  $\pi_S(x) = (x_{i_1}, \dots, x_{i_{|S|}})$  where  $i_j \in S$ . When the meaning is clear, we will write  $\pi_i(x)$  for  $\pi_{\{i\}}(x)$ .

We will denote  $Q(X_1, X_2, \dots, X_k)$  to be the space that is a disjoint union of each of the spaces  $X_S$ , for  $S \subseteq [k]$ . Formally,

$$Q(X_1, X_2, \dots, X_k) = \bigcup_{S \subseteq [k]} \{(x_{i_1}, \dots, x_{i_{|S|}}) : x_i \in X_i, S = \{i_1, \dots, i_{|S|}\}\}.$$

Let  $Y$  be any space and  $f : Q(X_1, \dots, X_k) \rightarrow Y$  be any function. Then, for a set  $S \subseteq [k]$ , we define  $f_S : X_S \rightarrow Y$  to be the restriction of  $f$  to only those inputs that occur in  $X_S$ . We will abuse notation by writing, for  $S \subseteq T$  and  $x \in T$ ,  $f_S(x)$  to mean  $f_S(\pi_S(x))$ .

Let  $\mathcal{G}$  be a collection of subsets of  $[k]$ , and let  $\bar{S}$  denote  $[k] \setminus S$  for all  $S \in \mathcal{G}$ . We will say that a function  $f$  defined on  $Q(X_1, X_2, \dots, X_k)$  is *deterministic with respect to  $\mathcal{G}$*  if for all  $S \in \mathcal{G}$  and for all  $x, y \in X_{[k]}$  we have that  $f(x) = f(y)$  whenever both  $f_S(x) = f_S(y)$  and  $f_{\bar{S}}(x) = f_{\bar{S}}(y)$ .

In essence the definition above is designed to capture the property of sumsets that was mentioned at the beginning of Section 4.2. For a function  $f$  to be deterministic with

respect to a single set  $S \subseteq [k]$ , it must be that  $f_S(x)$  and  $f_{\overline{S}}(x)$  uniquely determine the value of  $f(x)$ . Then being deterministic with respect to a collection  $\mathcal{G}$  is nothing more than being deterministic with respect to all  $G \in \mathcal{G}$ . The following examples show that both Cartesian products of sets and linear combinations of sets (and so, in particular, sumsets) are deterministic with respect to  $\mathcal{G}$  for any  $\mathcal{G}$ .

**Example 1.** Let  $V$  be a vector space over the reals with basis vectors  $\{v_1, \dots, v_k\}$ . Let  $X_1, \dots, X_k \subseteq \mathbb{R}$  and define  $f : Q(X_1, \dots, X_k) \rightarrow V$  such that  $f_S(x) = \sum_{i \in S} \pi_i(x) v_i$ . Then  $f$  is deterministic with respect to  $\mathcal{G}$  for all  $\mathcal{G} \subseteq [k]$ .

*Proof.* Let  $x \in X_T$  for some  $T \subseteq [k]$  and let  $G \in \mathcal{G}$ . Then

$$f(x) = \sum_{i \in T} \pi_i(x) v_i = \sum_{i \in (G \cap T)} \pi_i(x) v_i + \sum_{i \in (\overline{G} \cap T)} \pi_i(x) v_i = f_G(x) + f_{\overline{G}}(x).$$

Thus knowing  $f_G(x)$  and  $f_{\overline{G}}(x)$  uniquely determines  $f(x)$ . Since this is true for any  $G \in \mathcal{G}$ ,  $f$  is deterministic with respect to  $\mathcal{G}$ .  $\square$

**Example 2.** Let  $\mathcal{A}$  be an Abelian group and  $X_1, \dots, X_k \subseteq \mathcal{A}$  and let  $c_1, \dots, c_k \in \mathbb{Z}$ . Define  $f : Q(X_1, \dots, X_k) \rightarrow \mathcal{A}$  such that  $f_S(x) = \sum_{i \in S} c_i \pi_i(x)$ . Then  $f$  is deterministic with respect to  $\mathcal{G}$  for all  $\mathcal{G} \subseteq [k]$ .

*Proof.* The proof is identical to Example 1, only replacing  $v_i$  with  $c_i$ .  $\square$

#### 4.2.2 Fractional Subadditivity

Our goal is to prove Theorem 4.2.1, stated below. Note that Example 1, which shows that the characteristic function is deterministic, shows that Theorem 4.2.1 is, in fact, a generalization of normal subadditivity in the way that was mentioned at the end of Section 4.1.2.

**Theorem 4.2.1.** Let  $X_1, X_2, \dots, X_k$  be finite sets,  $\mathcal{G} = \{(\alpha_G, G)\}$  be a fractional covering of  $[k]$ , and  $f$  be a function on  $Q(X_1, \dots, X_k)$  that is deterministic with respect to  $\mathcal{G}$ . Then for any set  $R \subseteq f(X_{[k]})$ ,

$$|R| \leq \prod_{G \in \mathcal{G}} \left| f_G \left( f_{[k]}^{-1}(R) \right) \right|^{\alpha_G}$$

*Proof.* For  $A, B \in X_G$ , we define the order relation  $<_{lex}$  as  $A <_{lex} B$  if  $A$  comes before  $B$  in lexicographical order. Now let  $R$  be given, and for each  $r \in R$ , let  $x_r$  be the smallest element of  $f_{[k]}^{-1}(R) \subseteq X_{[k]}$  in lexicographical order, and let  $X^R = \{x_r : r \in R\}$ . Let  $Z$  be a random variable that chooses uniformly from the elements of  $X^R$ , and let  $Z_i = \pi_i(Z)$  for all  $i \in [k]$ . Then, by fractional additivity,

$$\log(|R|) = H(Z) = H(Z_1, \dots, Z_k) \leq \sum_{G \in \mathcal{G}} \alpha_G H(Z_G), \quad (1)$$

where  $Z_G = \{Z_{i_1}, \dots, Z_{i_{|G|}}\}$  for  $G = \{i_j\} \subseteq [k]$ . Hence  $Z_G = \pi_G(Z)$  and by the chain rule of entropy, for each  $G \in \mathcal{G}$ , we have that:

$$H(\pi_G(Z)|f_G(Z)) + H(f_G(Z)) = H(Z_G, f(Z_G)) = H(f_G(Z)|\pi_G(Z)) + H(\pi_G(Z)). \quad (2)$$

Here,  $H(f_G(Z)|\pi_G(Z)) = 0$  since  $f_G$  is a deterministic function, and so plugging in to the above equation gives:

$$\log(|R|) \leq \sum_{G \in \mathcal{G}} \alpha_G H(Z_G) = \sum_{G \in \mathcal{G}} \alpha_G \left( H(f_G(Z)) - H(\pi_G(Z)|f_G(Z)) \right).$$

The key observation is the following somewhat surprising claim, whose proof is more or less obvious; this is also the essence (in addition to Han's inequality) of the proof of Theorem 1.2 in [28].

**Claim:**  $H(\pi_G(Z)|f_G(Z)) = 0$  for all  $G \in \mathcal{G}$ .

It suffices to show that, for every  $G$ ,  $f_G$  is a one-to-one function when the domain is restricted to  $\pi_G(Z)$ . Assume, for the sake of contradiction, that  $f_G$  is not one-to-one. Then there are two elements  $a \neq b \in X_G$  such that  $f_G(a) = f_G(b)$  and both  $Pr(Z_G = a)$  and  $Pr(Z_G = b)$  are non-zero. Thus there must be "preimages"  $A, B \in X^R$  such that  $\pi_G(A) = a$  and  $\pi_G(B) = b$  and  $A \neq B$  (since otherwise  $a = b$ ).

Without loss of generality, let  $A <_{lex} B$  and consider  $b' = \pi_{\overline{G}}(B) \in X_{\overline{G}}$ . Let  $A' \in X_{[k]}$  be the vector

$$A'(i) = \begin{cases} a_i & \text{for } i \in G \\ b'(i) & \text{for } i \notin G. \end{cases}$$

Clearly  $A' <_{lex} B$ , and since

$$f_G(A') = f_G(a) = f_G(b) = f_G(B) \quad \text{and} \quad f_{\overline{G}}(A') = f_{\overline{G}}(b') = f_{\overline{G}}(B)$$

we have that  $f(A') = f(B)$ . This is a contradiction, however, since we constructed  $B$  to be the smallest such element in  $X_{[k]}$  in lexicographical order. Hence the claim follows.

Using the claim, it follows that Equation (2) reduces to  $H(\pi_G(Z)) = H(f_G(Z))$ . Plugging this into Equation (1) yields:

$$\log(|R|) \leq \sum_{G \in \mathcal{G}} \alpha_G H(f_G(Z)) \leq \sum_{G \in \mathcal{G}} \alpha_G \log(|f_G(X^R)|) \leq \sum_{G \in \mathcal{G}} \alpha_G \log\left(|f_G(f_{[k]}^{-1}(R))|\right)$$

where the last inequality is due to the fact that  $X^R \subseteq f_{[k]}^{-1}(R)$ , and so our claimed result is true.  $\square$

### 4.2.3 Sumset Corollaries

The following corollaries represent a slight variant on Theorem 4.2.1 in that they do not directly define  $R$ ; rather, they pick a subset  $S$  of the image of one of the subspaces, then lift the preimage of  $S$  up to the top space. Questions of this type were first asked in [28].

**Corollary 4.2.2.** *Let  $A, B_1, B_2, \dots, B_k \subseteq \mathbb{R}$  and define  $\overline{B_i} = B_1 + \dots + B_{i-1} + B_{i+1} + \dots + B_k$  for  $i = 1, \dots, k$  and  $B = B_1 + \dots + B_k$ . Then for any  $S \subseteq B$ , we have that*

$$|A + S|^k \leq |S| \prod_{i=1}^k |A + \overline{B_i}|$$

*Proof.* Set  $X_1 = A$  and  $X_i = B_{i-1}$  for  $i = 2, \dots, k+1$ , and  $X = X_1, \dots, X_{k+1}$ . By Example 2, the collection of functions  $f_S(x) = \sum_{i \in S} x_i$  is deterministic with respect to any  $\mathcal{G}$ . To ease notation, we will write  $g_i = f_{\overline{\{i\}}}$ . Note that  $S \subseteq g_1(X)$ , so let  $Q = g_1^{-1}(S)$ , and set  $R = \{f(a, b_1, \dots, b_k)\}$  for all  $a \in A$  and all  $(b_1, \dots, b_k) \in Q$ . Now choose

$$\mathcal{G} = \left\{ \left( \frac{1}{k}, \overline{\{i\}} \right) \right\}_{i \in [k+1]}.$$

By Theorem 4.2.1, we have that

$$|R| \leq \prod_{G \in \mathcal{G}} \left| f_G(f_{[k]}^{-1}(R)) \right|^{\alpha_G},$$

and we use that

$$g_1(f_{[k]}^{-1}(R)) = S \quad \text{and} \quad g_i(f_{[k]}^{-1}(R)) \subseteq A + \overline{B_{i-1}}$$

for  $i = 2, \dots, k+1$ . The inequality follows.  $\square$



**Corollary 4.2.3.** *Let  $A, B_1, B_2, \dots, B_k \subseteq \mathbb{R}$ . Then for any  $S \subseteq B$ , we have that*

$$|A + S|^k \leq |S|^{k-1} \prod_{i=1}^k |A + B_i|.$$

*Proof.* The proof is the same as Corollary 4.2.3, just with the different fractional covering:

$$\mathcal{G} = \left\{ \left( \frac{1}{k}, \{1, i\} \right) \right\}_{i \in [k+1]}.$$

□

**Remark.** Clearly, various other covering families yield similar corollaries. We mention these only because they offer direct generalizations of Theorem 1.5 in [28].

### 4.3 Further Research

While the results in this chapter provide an entire collection of inequalities for sumsets, a number of known inequalities have not yet been shown to be implied by this method (see [45]). It would be interesting to see if these inequalities can be deduced by the fractional subadditivity (or other properties) of entropy. In particular, recent work of Madiman [40] derived sumset inequalities from the *entropy power inequality*.

Another interesting direction of research would be to consider sumsets in the context of nonabelian groups. Our results do not immediately extend to nonabelian groups, but perhaps a more thorough analysis would give similar results.

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